

Electoral Campaigns as Dynamic Contests*

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Abstract

We model electoral campaigns as dynamic contests in which two office-motivated candidates allocate their budgets over time to affect their relative popularity (i.e. odds of winning), which evolves over time as a mean-reverting stochastic process. In each period of the model, the equilibrium ratio of spending by each candidate equals the ratio of their available budgets, and equilibrium spending is independent of the realizations of relative popularity. We extend the model to allow for (i) early voting, (ii) candidates who value money left over at the end of the race, (iii) multi-district competition, and (iv) endogenous budget processes that react to short-term fluctuations in the candidates' relative popularity. We also provide a theory of the electorate that serves as a foundation for the popularity process, and show that time-dependent regulations—for example, those that prohibit spending in the final stages of a campaign—can be welfare-enhancing and outperform static regulations—specifically, aggregate spending caps. Finally, we use the one-to-one relationship between the speed of reversion of the popularity process and the shape of the equilibrium spending path to recover estimates of the perceived rate of decay in popularity leads in actual elections. We use these estimates to examine the effects of dynamic regulations in races that include incumbents.

Key words: campaigns, dynamic allocation problems, contests

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1 Introduction

Electoral campaigns are dynamic contests in which strategic candidates allocate their campaign budgets across time in the run-up to the election. The contribution of this paper is to develop a tractable framework to study this dynamic allocation problem.

We focus on two-candidate races and assume that the candidates, 1 and 2, spend money to influence the movement of a state variable we call “relative popularity,” and eventually win the election, which takes place at a fixed date.¹ The two candidates start with one being possibly more popular than the other. At each moment in time, candidate 1’s popularity may increase relative to candidate 2’s; or it may decrease. Relative popularity evolves between periods (i.e. times in which the candidates make their spending decisions) according to a (possibly) mean-reverting Brownian motion. Period length is fixed, and the drift of relative popularity in a period depends on the spending decisions of the candidates in that period: it is strictly increasing and strictly quasiconcave in candidate 1’s spending and strictly decreasing and strictly quasiconvex in candidate 2’s. At the final date, an election takes place and the more popular candidate wins office.

Our baseline model is a zero-sum game in which the candidates are purely office-motivated and have a fixed budget. For this model, we show that the unique equilibrium path has two distinctive features. First, each candidate’s spending is independent of the relative popularity process. Second, an “equal spending ratio result” holds: at every history, the two candidates spend the same share of their remaining budgets.

For a parametric example of our baseline model, suppose that the long run mean of the popularity process is $\frac{1}{\lambda} \log(x/y)$ where x is candidate 1’s spending level, y is candidate 2’s, and λ is the speed of mean-reversion of the popularity process. Then, the equilibrium ratio of spending by either candidate in consecutive periods is constant over time and given by $e^{\lambda\Delta}$, where $\Delta > 0$ is the length of a period. In the limit as $\lambda \rightarrow 0$ (the case of no mean-reversion), the candidates spread their resources evenly across periods.² For $\lambda > 0$, both candidates increase their spending over time. As λ increases, they spend increasingly more towards the end of the race and less in the early stages.

The assumption of a mean-reverting popularity process is motivated by the empirical findings of Gerber et al. (2011), Hill et al. (2013) and others who highlight the *decay rate* in political advertising. In particular, these papers show that while money spent on campaign ads has a positive effect on support for the advertising candidate, these effects appear to decay rapidly over time.³ In our model, popularity leads decay toward the long run mean

¹A key premise of our model is that money spent (on advertising) can influence elections. For recent evidence on this, see Spenkuch and Toniatti (2018) and Martin (2014). For a summary of prior work on the effects of money spent in elections, see Jacobson (2015) and the discussion below on previous work in marketing and political advertising.

²This special case recovers the finding of a recent paper by Klumpp et al. (2019) that studies a strategic allocation problem absent the feature of decay; see our discussion below.

³See DellaVigna and Gentzkow (2010), Kalla and Broockman (2018), Jacobson (2015) and the references therein for work on the effects of political advertising, and persuasion more generally.

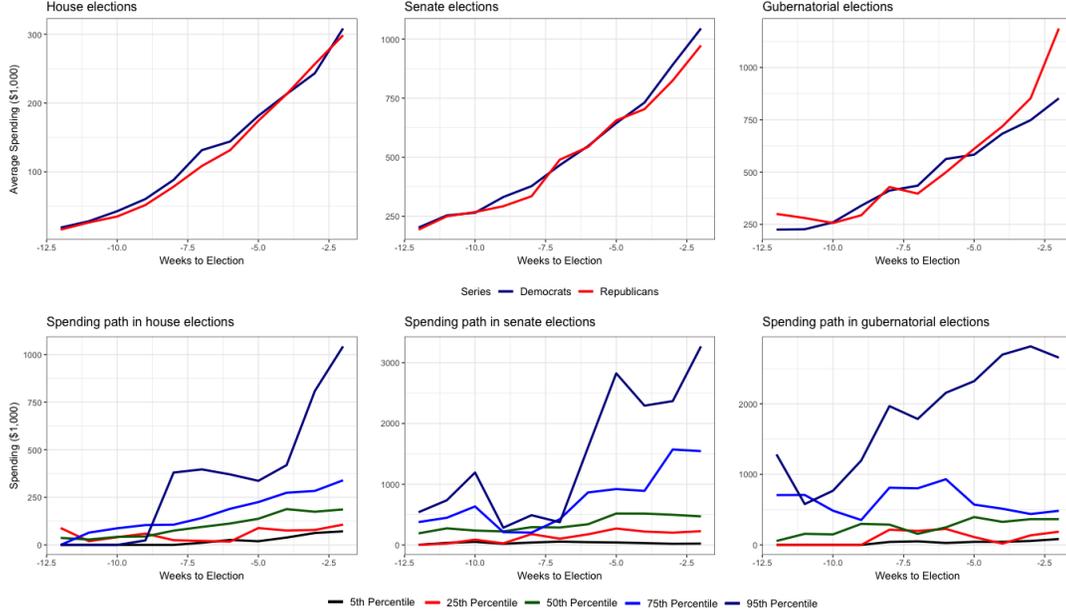


Figure 1: Upper figures are average spending paths by Democrats and Republicans on TV ads in “competitive” House, Senate and gubernatorial races in the period 2000-2014. These are elections in which both candidates spent a positive amount; see Section 5.1 for the source of these data, and more details. Bottom figures are spending paths for 5th, 25th, 50th, 75th, and 95th percentile candidates in terms of total money spent in the corresponding elections of the upper panel.

at rate $1 - e^{-\lambda\Delta}$, so fixing the length of a period, λ measures the decay rate: if $\lambda = 0$ there is no decay, and the higher is λ the higher is the per-period decay rate.

The case in which popularity leads decay over time rationalizes the pattern of spending in actual elections. Figure 1 shows the pattern of TV ad spending over time for candidates in U.S. House, Senate and gubernatorial elections over the period 2000-2014. The upper figures show that average spending for Democrats and Republicans in these races are nearly identical, and that average spending increases over time. The lower figures show that, despite the larger noise, the overall pattern of spending growth holds at the individual candidate level as well, especially in campaigns with the highest spending levels.

We also investigate other factors that determine the spending path. First, we provide an extension in which some voters turn out to vote early, several periods prior to the election date. We show that if more voters are expected to cast their ballots early, then the candidates save less for the end. Second, we relax the assumption that the game is zero-sum, and allow the candidates to value money that is unspent by the end of the race. Here, we assume (as in the parametric example above) that spending affects the long-term mean of the popularity process only through the ratio of the candidates’ spending levels in each period—an assumption that we call “ratio scale invariance.” Under this assumption,

we show that in every period the ratio of the candidates’ spending levels is constant and equal to the inverse ratio of their marginal values for money. However, in this extension, spending levels do vary with relative popularity: if the election is lopsided (in the sense that one candidate is more popular than the other), then both candidates spend less. Third, we look at an extension in which competition is over multiple districts (or media markets) and show that our equal-spending ratio result holds district-by-district. The fourth and final extension allows the candidates’ budgets to evolve stochastically over time in response to shifts in relative popularity. Here, candidates may have an added incentive to spend early in the race: early short-term popularity gains can help them raise more money that they can then deploy in the later stages of the race. The equal spending ratio result holds in all four extensions and facilitates a tractable analysis.

We also provide a model of the electorate to justify the mean-reverting popularity process in our baseline model. This model is based on the idea that a group of “impressionable” voters react to campaign advertising in the same way that the marketing literature has modelled product goodwill among consumers, following the seminal work of Nerlove and Arrow (1962).⁴ Providing a foundation for the electorate also enables us to study the welfare implications of campaign finance policies. For high decay rates, a welfare-enhancing policy is a “campaign silence” period that bans all spending in the final stages of a campaign. This policy yields a higher welfare than the laissez faire case of no regulations and it also yields a higher welfare than a static spending cap that places a bound on aggregate spending.

We end the paper with a closer look at patterns of campaign spending in the actual elections presented in Figure 1. Our key equal spending ratio result holds reasonably well in the data: in more than 80% of our dataset, the candidates’ spending ratios in a week are within ten percentage points of each other. This suggests that the model provides a good fit for spending in actual elections. We then fit our model to spending data to obtain estimates of the perceived decay rates in the popularity process.⁵ These perceived decay rates are an important quantity of interest in practice because they tell us how campaigns view a key factor that drives their spending decisions. They may also be useful as a benchmark for future campaigns seeking to optimize their spending. We show the use of our estimates by deriving the estimated effect of spending restrictions in the final periods of a campaign on the re-election rates of incumbents. Our results suggest that on average even just a one-week campaign ban brings incumbent re-election rates substantially closer to 50%.

Our paper relates to the prior literature on campaigning. Kawai and Sunada (2015), for instance, build on the work of Erikson and Palfrey (1993, 2000) to estimate a model of

⁴This formulation is also inspired by theory of impressionable voters developed in the political economy literature by Andonie and Diermeier (2019).

⁵Recall from the example above that there is a one-to-one relationship between the decay rate and the shape of candidates’ spending path: the higher is the decay rate, the greater is the speed of reversion, and the candidates spend more money towards the end of the race and less in the early stages. Therefore, we can recover the candidates’ perceived decay rate from actual spending data, without polling data which are generally too sparse to use in estimating decay rates with any reasonable precision.

fund-raising and campaigning. While Kawai and Sunada (2015) assumes that candidates allocate resources across different elections, we study the allocation problem across periods in the run-up to a particular election. de Roos and Sarafidis (2018) explain how candidates that have won past races may enjoy “momentum,” which results from a complementarity between prior successes and the current returns to spending.⁶

Meirowitz (2008) studies a static model to show how asymmetries in the cost of effort can explain the incumbency advantage. Polborn and David (2004) and Skaperdas and Grofman (1995) also examine static campaigning models in which candidates choose between positive or negative advertising.⁷ Iaryczower et al. (2017) estimate a model in which campaign spending weakens electoral accountability, assuming that the cost of spending is exogenous rather than subject to an inter-temporal budget constraint as in our model. Garcia-Jimeno and Yildirim (2017) estimate a dynamic model of campaigning in which candidates decide how to target voters taking into account the strategic role of the media in communication. Finally, Gul and Pesendorfer (2012) study a model of campaigning in which candidates provide information to voters over time, and face the strategic timing decision of when to stop. In our model, on the other hand, the end of the campaign is exogenous—the election day is fixed—and the candidates compete to affect the drift of the popularity process.

Our work is also related to the literature in marketing and operations research that analyzes advertising as a stochastic control problem. In the seminal work of Nerlove and Arrow (1962), an agent controls the “stock of goodwill” over time by continuously deciding how much to spend on advertising, while goodwill depreciates. More recently, Marinelli (2007) studies a problem similar to ours in a decision-theoretic framework, with a single advertiser facing an exogenous launch date for a product. The stock of goodwill is modeled as a Brownian motion that is controlled by spending. In the optimal control strategy the advertiser spends nothing until an intermediate time, and then she spends the maximum amount possible until the launch date.⁸

⁶Other dynamic models of electoral campaigns in which candidates enjoy momentum—such as Callander (2007), Knight and Schiff (2010), Ali and Kartik (2012)—are models of *sequential voting*. In contrast to this work, we abstract away from informational spillovers among voters in sequential elections, and focus on the candidates’ resource allocation problem within a single election.

⁷Other static models of campaigning include Prat (2002) and Coate (2004), who investigate how one-shot campaign advertising financed by interest groups can affect elections and voter welfare, and Krassa and Polborn (2010) who study a model in which candidates compete on the level of effort that they apply to different policy areas. Prato and Wolton (2018) study the effects of reputation and partisan imbalances on the electoral outcome. Compared to these papers, we study the dynamics of campaign spending and we characterize conditions under which spending is independent of the popularity history.

⁸Feichtinger et al. (1994) provide a survey of the literature on stochastic control models in advertising. A rich body of work studies dynamic advertising for regular consumer goods (in the absence of a product launch), where advertisers tend to use a ‘pulsing’ strategy: short, high-intensity periods of ad spending followed by no spending at all. This pattern of spending is justified through a threshold-based (Dubé et al., 2005) or an S-shaped sales response curve to advertising (Feinberg, 2001, Aravindakshan and Naik, 2015). In a related paper also using a model with a stock of goodwill that depreciates over time, Bronnenberg et al. (2012) study the longer-term effects of marketing and brand images.

The effects of advertising on elections is also studied in the marketing literature (see Gordon et al., 2012, for an early contribution). For example, Gordon and Hartmann (2013) estimate that political advertising impacts the outcome of U.S. presidential elections, although advertising elasticities are smaller than for other branded goods.⁹ Lovett and Peress (2015) estimate a model of targeted political advertising and find that TV ads mostly target swing voters. Chung and Zhang (2015) estimate the effectiveness of different campaign activities for the two major parties in U.S. presidential elections. Our model contributes to this literature by providing a tractable theoretical framework to study the allocation of advertising resources over time in a generic electoral competition setting.

In a game theoretic setting similar to ours, Kwon and Zhang (2015) study a two player model of stochastic control and strategic exit, motivated by a duopolistic market where market shares are modeled as a general diffusion process, and the firms can choose to exit at any time.¹⁰ Our paper also relates to Kamada and Kandori (2020) who apply their theory of revision games to electoral campaigning, and to Kamada and Sugaya (2020) who study electoral campaigning as a finite-horizon dynamic game.

Insofar as it studies campaigning as a dynamic strategic allocation problem, our paper relates to the vast literature on dynamic contests (see Konrad et al., 2009, and Vojnović, 2016 for reviews). Within this literature, Glazer and Hassin (2000) and Hinnesaar (2018) study contests in which multiple players move sequentially and only once, while we consider a setting in which the same two candidates move repeatedly over multiple periods.

In our model, competition unfolds stochastically over time. This relates our work to models of strategic races (see Harris and Vickers, 1985, 1987 for seminal contributions to this literature).¹¹ We differ from this literature both in terms of assumptions and in terms of methods. For instance, the decay of popularity leads distinguishes our paper from Klumpp and Polborn (2006), Konrad and Kovenock (2009) and Klumpp et al. (2019). From a methodological point of view, we characterize the outcome of the race leveraging the first order approach in a novel way. This enables us to characterize features of the spending path without actually solving the first order conditions.¹²

⁹Gordon and Hartmann (2016) also focus on U.S. presidential elections and find that the electoral colleges skews the allocations of advertising resources toward battleground states and increases overall spending when the election is not tight.

¹⁰Our focus on electoral campaigns differentiates us from this work. From a modeling standpoint, our approach in which two players simultaneously take actions in pre-determined periods makes our setup more tractable and it allows us to fully characterize the unique equilibrium path of spending.

¹¹Lee and Wilde (1980) and Reinganum (1981, 1982) study races over time in the presence of uncertainty, but their models do not cover situations in which one competitor leads or trails against the others.

¹²In particular, we show in Appendix B that the first order approach is valid when the players' payoffs are homothetic in their investments. This connects our paper to Cornes and Hartley (2005), Kolmar and Rommeswinkel (2013), Choi et al. (2016), Konishi et al. (2019) and Crutzen et al. (2020), who use CES functions in static contests to aggregate individual efforts. We differ from these papers as we study the allocation of a fixed resource over time. Thus, we have to take into account the possibility that players adjust future investments based on the realized impact of current investments.

The dynamic contest paper that is most closely related to our is Klumpp et al. (2019), who also study a dynamic strategic allocation model and find that absent any decay, the allocation of resources over time is constant. Our work builds on theirs by uncovering the fact that the equal spending ratio result holds in a variety of general settings that are motivated by our application to electoral campaigning.

2 Baseline Model

2.1 The Dynamic Campaigning Game

Consider the following complete information *dynamic campaigning game* between two candidates, $i = 1, 2$, ahead of an election. Time runs continuously from 0 to T and candidates take actions at times in $\mathcal{T} := \{0, \Delta, 2\Delta, \dots, (N-1)\Delta\}$, with $\Delta := T/N$ being the interval between consecutive actions. We identify these times with N discrete periods indexed by $n \in \{0, \dots, N-1\}$. For all $t \in [0, T]$, we use $\underline{t} := \max\{\tau \in \mathcal{T} : \tau \leq t\}$ to denote the last time that the candidates took actions.

At the start of the game the candidates are endowed with resource stocks, $X_0 > 0$ and $Y_0 > 0$ respectively for candidates 1 and 2.¹³ Candidates allocate their resources across periods to influence changes in their *relative popularity*. Relative popularity at time t is measured by a continuous random variable $Z_t \in \mathbb{R}$ whose realization at time t is denoted z_t . We interpret this as a measure of candidate 1's lead in the polls. If $z_t > 0$, then candidate 1 is ahead of candidate 2. If $z_t < 0$, then candidate 2 is ahead; and if $z_t = 0$, it is a dead heat. At the beginning of the game, relative popularity is equal to some $z_0 \in \mathbb{R}$.

At every time $t \in \mathcal{T}$, the candidates simultaneously decide how much of their resource stock to invest in influencing their future relative popularity. Candidate 1's investment is denoted x_t while candidate 2's is denoted y_t . The size of the resource stock that is available to candidate 1 at time $t \in \mathcal{T}$ is $X_t = X_0 - \sum_{\tau \in \{t' \in \mathcal{T} : t' < t\}} x_\tau$ and the size of the resource stock available to candidate 2 at time $t \in \mathcal{T}$ is $Y_t = Y_0 - \sum_{\tau \in \{t' \in \mathcal{T} : t' < t\}} y_\tau$. At every time $t \in \mathcal{T}$, budget constraints must be satisfied, so $x_t \leq X_t$ and $y_t \leq Y_t$.

Throughout, we maintain the assumption that for all times t , the evolution of popularity is governed by the following Brownian motion:

$$dZ_t = (p(x_t, y_t) - \lambda Z_t) dt + \sigma dW_t \tag{1}$$

where W_t is the Wiener process, $\lambda \geq 0$ and $\sigma > 0$ are parameters, and $p(\cdot)$ is a twice differentiable real-valued function. The drift of popularity therefore depends on the candidates'

¹³Although candidates raise funds over time, our assumption that they start with a fixed stock is tantamount to assuming that they can forecast how much will be available to them. In fact, some large donors make pledges early on and disburse their funds as they are needed over time. Nevertheless, in Section 3.4 we relax this assumption and consider an extension of the model in which the candidates' resources evolve over time in response to their relative popularity.

investments through the function $p(\cdot)$, which we assume to be strictly increasing in x and strictly decreasing in y . If $\lambda = 0$ the law of motion of relative popularity in the interval between consecutive periods of investment, t and $t + \Delta$, is a standard Brownian motion with drift $p(x_t, y_t)$. If $\lambda > 0$, it is a mean reverting Brownian motion (the Ornstein-Uhlenbeck process) with long-run mean $p(x_t, y_t)/\lambda$ and speed of reversion λ .

The game ends at time T , with candidate 1 winning if $z_T > 0$, losing if $z_T < 0$, and both candidates winning with equal probability if $z_T = 0$. The winner collects a payoff of 1 while the loser collects a payoff of 0. Thus, the game is zero sum, and the winner is the candidate that is more popular at time T .

Interpretations. The model is general enough to account for factors such as incumbency advantage, past legislative record, candidate valence, or name recognition, to the extent that their effects can be modelled through the initial lead in relative popularity, z_0 , relative resource endowments, X_0/Y_0 , or any asymmetry across the two candidates in the long-run mean of the popularity process, $p(x, y)/\lambda$. Through the derivatives of p , we can also capture differences in the efficacy of campaign spending between candidates, due to differences in the setup of the campaigns, to one candidate being better at campaigning than the other, or to one candidate's policy platform being more popular than the other's.

2.2 Equilibrium Analysis

Since the popularity process is a continuous-time process, strategies must be measurable with respect to the filtration generated by W_t . However, since candidates take actions only at discrete times, we will forgo this additional formalism and treat the game as a game in discrete time.

By our assumption about the popularity process in (1), the distribution of $Z_{t+\Delta}$ at any time $t \in \mathcal{T}$, conditional on (x_t, y_t, z_t) , is normal with constant variance and a mean that is a weighted sum of $p(x_t, y_t)$ and z_t ; specifically,

$$Z_{t+\Delta} \mid (x_t, y_t, z_t) \sim \begin{cases} \mathcal{N}(p(x_t, y_t)\Delta + z_t, \sigma^2\Delta) & \text{if } \lambda = 0 \\ \mathcal{N}((1 - e^{-\lambda\Delta})p(x_t, y_t)/\lambda + e^{-\lambda\Delta}z_t, \sigma^2(1 - e^{-2\lambda\Delta})/2\lambda) & \text{if } \lambda > 0 \end{cases} \quad (2)$$

where $\mathcal{N}(\cdot, \cdot)$ denotes the normal distribution whose first component is mean and second is variance. Note that the mean and variance of $Z_{t+\Delta}$ in the $\lambda = 0$ case correspond to the limits as $\lambda \rightarrow 0$ of the mean and variance in the $\lambda > 0$ case.

The model is therefore strategically equivalent to a discrete time model in which relative popularity is a state variable that transitions over discrete periods, and in each period it is normally distributed with a constant variance and a mean that depends on the popularity in the last period and on the candidates' spending levels.

The key implication of (2) is that the effect of the spending levels on the next period popularity level is linearly separable from the stochastic terms (Z_t, ε_t) , which we can see by

writing for all $t \in \mathcal{T}$,

$$Z_{t+\Delta} = (1 - e^{-\lambda\Delta})p(x_t, y_t) + e^{-\lambda\Delta}Z_t + \varepsilon_t, \quad (3)$$

where ε_t is a mean-zero normally distributed random variable.¹⁴ By recursive substitution we can write

$$Z_T = (1 - e^{-\lambda\Delta}) \sum_{n=0}^{N-1} e^{-\lambda\Delta(N-1-n)} p(x_{n\Delta}, y_{n\Delta}) + z_0 e^{-\lambda N\Delta} + \sum_{n=0}^{N-1} e^{-\lambda\Delta(N-1-n)} \varepsilon_{n\Delta} \quad (4)$$

where $(\varepsilon_\tau)_{\tau \geq 0}$ are i.i.d. normal shocks all with mean 0. Candidate 1 maximizes $\Pr[Z_T > 0]$ and candidate 2 minimizes this probability. Since the coefficients of the ε terms in (4) do not depend on $x_{n\Delta}$ and $y_{n\Delta}$, the variance of Z_T is independent of the candidates' strategies. So we can hereafter write the objective of candidate 1 as maximizing the expected value of Z_T and the objective of candidate 2 as minimizing it.

We say that a pure strategy subgame perfect equilibrium (SPE) of the game is *interior* if the first order conditions for the candidates' maximization problems are satisfied. Our equilibrium concept throughout is interior pure strategy SPE, which we refer to succinctly as "equilibrium."¹⁵ Sufficient conditions for a unique equilibrium to exist are that $p(\cdot, y)$ is strictly quasiconcave for all y , $p(x, \cdot)$ is strictly quasiconvex for all x , and the following *Inada-0* conditions hold: $\lim_{x \rightarrow 0} \partial p(x, y) / \partial x = \infty$ for all y and $\lim_{y \rightarrow 0} \partial p(x, y) / \partial y = -\infty$ for all x . We assume throughout that all of these sufficient conditions hold.

We now make some simple observations about the equilibrium of the game. These observations, which we summarize in Proposition 1, follow immediately from (4) and known results in the literature. First, the spending profile (x_t, y_t) is independent of $(z_\tau)_{\tau \leq t}$ because the shock terms are separable from the spending choices in (4). Second, since spending decisions do not depend on the history of the popularity process, the path of spending is the same even if we limit the extent to which this process is observable or can be inferred by the candidates. Finally, because equation (4) implies that a candidate's spending decision is independent of the other candidate's past spending, and the game is zero-sum, the Nash equilibrium path of play is unique and robust to allowing the candidates to move sequentially with arbitrary order of moves within a period (see, e.g., Mertens et al., 2015).

Proposition 1. *The equilibrium of the dynamic campaigning game is unique and*

- (i) *independent of the past history $(z_\tau)_{\tau \leq t}$ of relative popularity, and has the same path of play as any equilibrium of the alternative version of the game where candidates imperfectly and asymmetrically observe the realization of the path of popularity.*

¹⁴Using the result in Karatzas and Shreve (1998) equation (6.30), we can write down sufficient conditions to obtain this separability. Details are available upon request.

¹⁵Because the game that we consider is zero-sum, for any Nash equilibrium there exists an outcome-equivalent SPE. So we will sometimes look at Nash equilibria to study on-path equilibrium play.

(ii) has the same path of play as every Nash equilibrium of the game, and any equilibrium of the alternative version of the game where candidates move sequentially within a period, with arbitrary (and possibly stochastic) order of moves across periods.

2.3 Solving for the Spending Path

To solve for the optimal spending path, we must put some additional structure on the function $p(x, y)$. Suppose that $p(x, y) = f(\alpha_1\varphi(x) - \alpha_2\varphi(y))$ where f is a twice differentiable and increasing function, α_1 and α_2 are constants, and φ is a function such that $\varphi'(x) = x^\beta$ for some parameter β . Let us assume that these functions and parameters are chosen so that a unique equilibrium exists.¹⁶

Here, and throughout the paper, we will refer to the ratio of a candidate's current spending to current budget as that candidate's *spending ratio*. For candidate 1 this is x_t/X_t and for candidate 2 it is y_t/Y_t for all $t \in \mathcal{T}$. We will also refer to a candidate's *consecutive period spending ratio* as the ratio of next period spending to current period spending. For candidate 1, this is $x_{t+\Delta}/x_t$ and for candidate 2 it is $y_{t+\Delta}/y_t$.

Given Z_T from (4), at any time $t \in \mathcal{T}$ candidate 1 maximizes $\Pr[Z_T \geq 0 \mid z_t, X_t, Y_t]$ under the constraint $\sum_{n=t/\Delta}^{N-1} x_{n\Delta} \leq X_t$, while candidate 2 minimizes this probability under the constraint $\sum_{n=t/\Delta}^{N-1} y_{n\Delta} \leq Y_t$. Using this fact, we can apply the Euler method from consumer theory to solve the equilibrium, provided the first order conditions are sufficient and f is a homogeneous function of degree 1.

The candidates' first order conditions with respect to $x_{n\Delta}$ and $y_{n\Delta}$ for each $n < N - 1$ are respectively:

$$\begin{aligned} e^{-\lambda\Delta(N-1-n)} x_{n\Delta}^\beta f'(\alpha_1\varphi(x_{n\Delta}) - \alpha_2\varphi(y_{n\Delta})) &= x_{(N-1)\Delta}^\beta f'(\alpha_1\varphi(x_{(N-1)\Delta}) - \alpha_2\varphi(y_{(N-1)\Delta})), \\ e^{-\lambda\Delta(N-1-n)} y_{n\Delta}^\beta f'(\alpha_1\varphi(x_{n\Delta}) - \alpha_2\varphi(y_{n\Delta})) &= y_{(N-1)\Delta}^\beta f'(\alpha_1\varphi(x_{(N-1)\Delta}) - \alpha_2\varphi(y_{(N-1)\Delta})). \end{aligned}$$

¹⁶If f is the identity function, for example, an equilibrium exists when $\beta < 0$ and $\alpha_1, \alpha_2 > 0$. The example mentioned in the introduction is a special case in which $\alpha_1 = \alpha_2 = -\beta = 1$ so that $p(x, y) = \log(x/y)$. In this case, the model is not closed because p is undefined when $y = 0$. To get around this, we assume that if either candidate i spends 0 at any time in \mathcal{T} , then the game ends immediately. If candidate $j \neq i$ spends a positive amount at that time, then j is the winner while if j also spends 0 at that time, then each candidate wins with probability 1/2. This ensures that in the continuous time setting, Z_t will follow an Itô process at every history. Moreover, the model can be considered the limiting case of two different models. One is a model in which the marginal return to spending an ϵ amount of resources starting at 0 goes to infinity. The other is one in which candidates have to spend a minimum amount ϵ in each period to sustain the campaign, and ϵ goes to 0. However, with these assumptions we will have to say that equilibria are "essentially" unique since there is a trivial source of multiplicity that arises at histories in which one candidate spends 0 in a given period (though these histories do not arise on the equilibrium path). In this case, if the other candidate has a positive resource stock, he may spend any positive amount in that period and win. Apart from this kind of multiplicity, equilibria will be unique. We take this approach throughout the paper whenever we encounter the same dividing-by-zero problem.

By taking the ratio of these conditions, we see that for all n , $x_{n\Delta}/y_{n\Delta} = x_{(N-1)\Delta}/y_{(N-1)\Delta}$. Therefore, using the candidates' budget constraints, we get that for all periods n

$$\frac{x_{n\Delta}}{X_{n\Delta}} = \frac{y_{n\Delta}}{Y_{n\Delta}}.$$

We refer to this result as the “equal spending ratio result.” The equal spending ratio result has two important implications: (i) the ratio of candidates nominal spending is constant over time, $x_{n\Delta}/y_{n\Delta} = X_0/Y_0$ for every period n , and (ii) the consecutive period spending ratio is the same on the equilibrium path for both candidates:

$$r_{n\Delta} := \frac{x_{(n+1)\Delta}}{x_{n\Delta}} = \frac{y_{(n+1)\Delta}}{y_{n\Delta}}.$$

To find the equilibrium, we will guess that the common consecutive period spending ratio is constant over time, and then verify this guess. Let $r_{n\Delta} = r$ for all periods n . If we equate the left hand sides of candidate 1's first order conditions for two consecutive periods n and $n + 1$ we get

$$e^{-\lambda\Delta} x_{n\Delta}^\beta f'(\alpha_1\varphi(x_{n\Delta}) - \alpha_2\varphi(y_{n\Delta})) = x_{(n+1)\Delta}^\beta f'(\alpha_1\varphi(x_{(n+1)\Delta}) - \alpha_2\varphi(y_{(n+1)\Delta})) \quad (5)$$

If the guess of constant spending growth is correct then

$$\begin{aligned} f'(\alpha_1\varphi(x_{(n+1)\Delta}) - \alpha_2\varphi(y_{(n+1)\Delta})) &= f'(r^{1+\beta}(\alpha_1\varphi(x_{n\Delta}) - \alpha_2\varphi(y_{n\Delta}))) \\ &= f'(\alpha_1\varphi(x_{n\Delta}) - \alpha_2\varphi(y_{n\Delta})) \end{aligned}$$

since $\varphi(x) = x^{1+\beta}/(1 + \beta)$ and the derivative of a homogeneous function of degree 1 is a homogeneous function of degree 0. Therefore, using this in equation (5), we get that $r = \exp(-\lambda\Delta/\beta)$. The same holds for candidate 2. This fully characterizes the equilibrium spending path, and verifies our guess that the consecutive period spending ratio is constant over time; we refer to this as the “constant spending growth result.”

The next proposition summarizes the findings.¹⁷

Proposition 2. *The equilibrium spending path $(x_t, y_t)_{t \in \mathcal{T}}$ solved for the specification of the baseline model in this section*

- (i) *satisfies the equal spending ratio result: $x_t/X_t = y_t/Y_t$ for all $t \in \mathcal{T}$,*
- (ii) *exhibits constant spending growth: $r_t = r = \exp(-\lambda\Delta/\beta)$ for all $t \in \mathcal{T}$.*

¹⁷Provided that the first order conditions are sufficient, we could let f be homogeneous of degree d for arbitrary $d \geq 1$. In this case, f' is homogenous of degree $d - 1$ and the result of Proposition 2 holds with

$$r = \exp\left(-\frac{\lambda\Delta}{(1 + \beta)d - 1}\right).$$

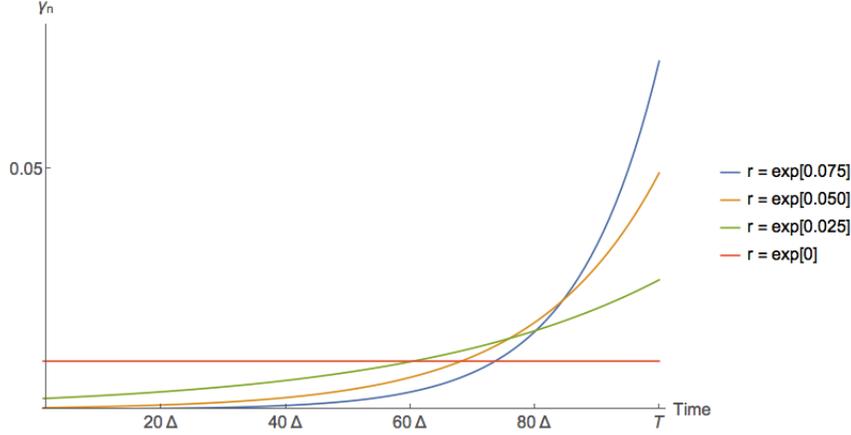


Figure 2: The fraction γ_n of initial budget that the candidates spend over time, for $N = 100$ and various values of r .

With fixed budgets, the ratio of consecutive period spending is sufficient to fully characterize the path of spending over time as mentioned in the proposition. For example, consider a benchmark case where f is the identity, and $\alpha_1, \alpha_2, -\beta > 0$ so that the assumptions for an interior equilibrium are satisfied. If $\lambda = 0$, meaning that popularity leads do not decay, then $r = 1$: in each period, the candidates spend a fraction $1/N$ of their initial budget. If $\lambda > 0$, then spending increases over time, and the fraction of the initial budget each candidate spends in period n is equal to

$$\gamma_n = \frac{x_n}{X_0} = \frac{y_n}{Y_0} = \frac{r - 1}{r^N - 1} r^n.$$

Substituting r from Proposition 2, we can derive the comparative statics of γ_n with respect to the parameters. For example, if β increases, the marginal return to spending diminishes at a slower rate. This gives candidates less of an incentive to smooth their spending over time. So, they spend more towards the end.¹⁸ As λ increases, the marginal benefit of spending early drops: any popularity advantage produced by an early investment has a tendency to decay and this tendency increases with λ . Candidates thus have an incentive to invest less in the early stages and more in the later stages of the race. Figure 2 depicts these features by plotting γ_n for different values of r .

These results and those of the previous section extend beyond the Brownian structure that we assume. As part of a general analysis of dynamic two-players contests with fixed

¹⁸In fact, as $\beta \rightarrow 0^-$ the marginal return to spending does not diminish and candidates spend all of their resources in the final period.

resource stocks and a stochastic state, Appendix B provides general conditions under which these results hold, as well as some additional intuition.

3 Extensions

In this section, we extend our baseline model to capture important features of electoral campaigns.

3.1 Early Voting

In many elections, voters are able to cast their votes prior to election day, either by mail or in person. Our model is able to accommodate this kind of early voting.

Consider the setting of Section 2.3, but now suppose that voters can vote from time $\hat{N}\Delta$ onwards, where $\hat{N} < N$ is an integer. Furthermore, suppose that the vote difference among votes cast for the two candidates in each period $n \geq \hat{N}$ is proportional to the relative popularity $Z_{n\Delta}$ in that period. Finally, let the number of total votes cast in period $n \geq \hat{N}$ be a proportion $\xi \in (0, 1)$ of the total votes cast in period $n + 1$. Therefore, the higher is ξ , the lower is the growth rate in votes cast as election day approaches, and if ξ is close to zero, then almost all votes are cast at time T . Candidate 1 thus maximizes (candidate 2 minimizes):

$$\Pr \left[\sum_{k=0}^{N-\hat{N}} \xi^k Z_{(N-k)\Delta} \geq 0 \right].$$

Let us also assume that, despite early voting, either candidate can win the election if his popularity at time T is sufficiently high, no matter how low it was in previous periods.¹⁹

Proposition 3. *In an equilibrium of this early voting extension, the equal spending ratio result holds: if $X_t, Y_t > 0$ then $x_t/X_t = y_t/Y_t$ for all $t \in \mathcal{T}$. In addition, the consecutive period spending ratio for both candidates is*

$$\hat{r}_n = \begin{cases} e^{-\frac{\lambda\Delta}{\beta}} & \text{if } n < \hat{N} \\ K(\xi, \lambda\Delta, \beta)e^{-\frac{\lambda\Delta}{\beta}} & \text{if } n \geq \hat{N} \end{cases}$$

where

$$K(\xi, \lambda\Delta, \beta) := \left(\frac{(e^{-\lambda\Delta/\xi}) - (e^{-\lambda\Delta/\xi})^{N-n-1}}{1 - (e^{-\lambda\Delta/\xi})^{N-n-1}} \right)^{-\frac{1}{\beta}}.$$

$K(\xi, \lambda\Delta, \beta)$ is lower than 1 and decreasing in n and ξ .

¹⁹This holds if $\xi(2 - \xi^{N-\hat{N}}) < 1$, which is implied by $\xi < 1/2$. Alternatively, we could assume that candidate 1 maximizes (and candidate 2 minimizes) the difference in candidate 1 and 2's vote share, $\sum_{k=0}^{N-\hat{N}} \xi^k Z_{(N-k)\Delta}$. The results of Proposition 3 would extend to this case.

Besides establishing the equal spending ratio result in this setting, Proposition 3 has two main implications. First, the consecutive period spending ratio is no longer constant over time. Second, as the early voting turnout rate increases (i.e., ξ increases), spending patterns become more evenly distributed over time. This extension thus highlights that with early voting, the dynamic pattern of spending is determined by two countervailing forces: the decay rate in popularity leads, measured by λ , pushes candidates to spend more resources toward the end of the race, while early voting turnout, measured by ξ , gives them an incentive to spend more resources in the earlier stages of the race.

3.2 Popularity Dependence when Candidates Value Leftover Money

In the baseline model candidates are purely office-motivated. This implies that both candidates fully deplete their budgets by the end of the campaign and the opportunity cost of spending in each period $t \in \mathcal{T}$ is determined by the value of spending in subsequent periods. However, money used in a campaign may have an outside value for candidates—e.g., a candidate may save money for future political endeavors or for investment opportunities outside politics (to the extent this is legally allowed).

To capture this possibility, let X_T and Y_T be the budgets leftover at the end of the campaign for candidates 1 and 2 respectively, and assume that candidate 1 maximizes $\Pr[Z_T \geq 0] + \kappa_1 X_T$, while candidate 2 maximizes $(1 - \Pr[Z_T \geq 0]) + \kappa_2 Y_T$. The parameter $\kappa_i > 0$, $i = 1, 2$, captures candidate i 's marginal value for money. On top of saving money and benefiting from this at rate κ_i , we also assume that each candidate i can overspend his budget by borrowing money paying a cost κ_i .²⁰ Hence X_T and Y_T can be negative.

All other features of the baseline model hold with one exception: candidates' spending choices affect the popularity process only through their ratio: $p(x, y) = q(x/y)$ for some strictly increasing and strictly quasiconcave function q . We refer to this assumption as the *ratio scale invariance* assumption.²¹

In this setting, the equilibrium ratio of candidates' nominal spending, x_t/y_t , is constant and equal to κ_2/κ_1 for all $t \in \mathcal{T}$. (Recall that in the baseline model this ratio is constant and equal to X_0/Y_0 .) Although the ratio of spending is constant over time, spending levels are not and depend on relative popularity. Spending by both candidates decreases as the race becomes more lopsided in terms of relative popularity. To show this, define the following quantity for all $n = \{0, \dots, N - 1\}$:

$$\zeta((\varepsilon_{m\Delta})_{m=0}^{n-1}) = \frac{\sum_{m=0}^{N-1} e^{-\lambda\Delta(N-1-m)} q(\kappa_1/\kappa_2) + z_0 e^{-\lambda N\Delta} + \sum_{m=0}^{n-1} e^{-\lambda\Delta(N-1-m)} \varepsilon_{m\Delta}}{\sigma \sqrt{\Delta} \sum_{m=n}^{N-1} e^{-2\lambda\Delta(N-1-m)}} \quad (6)$$

²⁰To simplify the analysis, we abstract from the time dimension when we model borrowing. In other words money borrowed at the beginning of the campaign or at the end of it entails the same cost κ_i .

²¹Ratio scale invariance is a strengthening of Assumption B.3 discussed in Appendix B.2.

where $\varepsilon_{m\Delta}$ is the aggregate popularity shock in period m . This quantity measures the expected electoral advantage of one candidate over the other as evaluated at time $t = n\Delta$. In a lopsided election, where one candidate has a large popularity advantage over the other, $\zeta((\varepsilon_{m\Delta})_{m=0}^{n-1})$ is large in absolute value. We then have the following result.

Proposition 4. *In an equilibrium of this extension, for all $n = 0, 1, \dots, N-1$, (i) $x_{n\Delta}/y_{n\Delta} = \kappa_2/\kappa_1$, and (ii) $x_{n\Delta}$ and $y_{n\Delta}$ are both decreasing in $|\zeta((\varepsilon_{m\Delta})_{m=0}^{n-1})|$.*

Part (i) of the proposition follows because, in equilibrium, the opportunity cost of spending is now given by the value from not spending in the campaign (whereas in the baseline model the opportunity cost is the value from being able to spend in some future period). Part (ii) of the proposition states that the candidates' spending levels are no longer independent of relative popularity: spending by both candidates increases as the election becomes more competitive. Examining the expression in (6), if the marginal values of money κ_1 and κ_2 increase proportionally for the two candidates, spending goes down uniformly in each period. Spending also reacts more to changes in relative popularity. Therefore, high-stakes elections—those for which κ_1 and κ_2 are small—should see overall higher levels of spending, and spending in these elections should be less responsive to fluctuations in relative popularity.

3.3 Multi-district Competition

In addition to the decision of when to spend, candidates also make decisions about where to spend their money. Suppose that the two candidates compete in \mathcal{S} districts or media markets rather than a single district or market, and the payoffs of the candidates depend on how these contests aggregate. With an arbitrary aggregation rule, this setting covers the electoral college for U.S. presidential elections, as well as competition between two parties seeking to control a majoritarian legislature composed of representatives from winner-take-all single-member districts, and even the case where candidates compete in a single winner-take-all race but they must choose how to allocate spending across different geographic media markets in the district.

Relative popularity in each district s is the random variable Z_t^s with realizations z_t^s . We assume that $(Z_t^s)_{s \in \mathcal{S}}$ follows a multidimensional Brownian motion with diffusion $\Sigma_{\mathcal{S}}$, where we allow for arbitrary correlation of popularity across districts: in each district s , the popularity process is

$$dZ_t^s = (p(x_t^s, y_t^s) - \lambda^s Z_t^s) dt + \sigma^s dW_t^s, \quad (7)$$

so that each district has its own speed of reversion λ^s , and its own diffusion σ^s applied to the district's Wiener process W_t^s . In addition, let us extend the ratio scale invariance assumption so that $p(x_t^s, y_t^s) = q(x_t^s/y_t^s)$ for some strictly increasing concave differential

function q .²² The aggregation rule for the outcomes in the various districts is arbitrary, but we impose a monotonicity requirement: in each period $t \in \mathcal{T}$ candidate 1's utility is strictly increasing in each Z_T^s , while candidate 2's utility is strictly decreasing in each Z_T^s . If all other structural features are the same as in the baseline model, then in equilibrium the equal spending ratio result holds district-by-district.

Proposition 5. *In an equilibrium of this multi-district extension, if $X_t, Y_t > 0$ are the candidates' remaining budgets at any time $t \in \mathcal{T}$, then for all districts s , $x_t^s/X_t = y_t^s/Y_t$.*

As a consequence, total spending by each of the two candidates across all districts at a given time also respects the equal spending ratio result: if $x_t = \sum_s x_t^s$ is candidate 1's total spending at time t and $y_t = \sum_s y_t^s$ is candidate 2's, then $x_t/X_t = y_t/Y_t$, for all $t \in \mathcal{T}$.

In this extension, unlike the baseline model, it may be the case that spending decisions depend on the popularity processes. If the competition in some districts becomes lopsided (in terms of the candidates' relative popularity), the marginal benefit of spending money in those districts for both candidates may decrease and the candidates may concentrate their spending in other more competitive districts. This means that relative popularity within districts plays a role in the spending decisions.

A special case in which this popularity dependence does not arise is the one in which the payoffs to the two candidates are a weighted sum of relative popularity in each district at time T . Let the weight of district s be $w^s > 0$. Without loss of generality, assume $\sum_{s=1}^S w^s = 1$. These assumptions would fit a media-market setting, where candidates allocate resources across media markets, or if the "candidates" are two parties that compete to maximize the number of seats in a legislature, seats are allocated proportionally in each district and the number of seats attributed to each district depends on the district population, which is reflected in w^s . In this case, the candidates' payoffs are:

$$u_1((Z_T^s)_{s=1}^S) = -u_2((Z_T^s)_{s=1}^S) = \sum_{s=1}^S w^s Z_T^s. \quad (8)$$

The following function measures the candidates' marginal value of spending at time $t = n\Delta$ in district s versus some other district s' :

$$\Lambda(\lambda^s, \lambda^{s'}, n) = \frac{(1 - e^{-\lambda^s \Delta}) \lambda^{s'}}{(1 - e^{-\lambda^{s'} \Delta}) \lambda^s} e^{-(N-1-n)(\lambda^s - \lambda^{s'}) \Delta} \quad (9)$$

Note that $\Lambda(\lambda^s, \lambda^{s'}, n) \geq 1$ if and only if $\lambda^{s'} \geq \lambda^s$, and $|\Lambda(\lambda^{s'}, \lambda^s, n) - 1|$ decreases with n . Using this function, the next proposition shows that the allocation of resources across districts is independent of the popularity processes and candidates spend overall more re-

²²We also extend the assumption in footnote 16 as follows: if a candidate spends an amount equal to 0 in any district, then the game ends and the candidate wins with probability 1/2 if the other candidate is also spending an amount equal to 0 in some district, and loses with probability 1 otherwise.

sources in districts that get greater weight, and have lower decay rates. The proposition also states that the differences in spending due to differences in decay rates are maximal at the beginning of the campaign and decrease as the election date approaches.²³

Proposition 6. *Suppose in this multi-district extension that the candidates' payoffs are given by (8). Then in equilibrium, for all $n = 0, \dots, N - 1$ and any pair of districts s, s' ,*

$$\frac{x_{n\Delta}^s}{x_{n\Delta}^{s'}} = \frac{y_{n\Delta}^s}{y_{n\Delta}^{s'}} = \frac{w^s}{w^{s'}} \Lambda(\lambda^s, \lambda^{s'}, n).$$

3.4 Evolving Budgets

Our baseline model assumes that candidates are endowed with a fixed budget at the start of the game (or they can perfectly forecast how much money they will raise), but in reality the amount of money the candidates can expect to raise is uncertain and this expectation evolves over time with changes in relative popularity. We now present an extension to account for this. (See the end of this section, for an alternative specification.) We retain all the features of the baseline model except the ones described below.

Candidates start with exogenous budgets X_0 and Y_0 as in the baseline model. However, we now assume that budgets evolve according to the following geometric Brownian motions:

$$\begin{aligned} \frac{dX_t}{X_t} &= az_t dt + \sigma_X dW_t^X & \text{if } X_t > 0; \\ \frac{dY_t}{Y_t} &= bz_t dt + \sigma_Y dW_t^Y & \text{if } Y_t > 0. \end{aligned}$$

a, b, σ_X and σ_Y are constants, and W_t^X and W_t^Y are Wiener processes. None of our results hinge on it, but for simplicity we assume that W_t is independent of W_t^X and of W_t^Y , while W_t^X and W_t^Y may covary. We also assume that if a candidate's budget reaches 0 at any moment in time, it is 0 thereafter: if $X_t = 0$ ($Y_t = 0$), then $X_\tau = 0$ ($Y_\tau = 0$) for all $\tau \geq t$.

In this setting, if $b < 0 < a$ then donors raise their support for the candidate that is leading in the polls and withdraw support from the one that is trailing. If $a < 0 < b$ then donors channel their resources to the underdog. Popularity therefore feeds back into the budget process. The feedback is positive if $a - b > 0$ and negative if $a - b < 0$. We refer to a and b as the feedback parameters.²⁴

²³The same result would hold even if we allow candidates' investments to impact the popularity process through a generic function $p(x, y)$ satisfying Assumption B.3. It is also possible to encompass the case in which the spending in some districts affects the popularity in others.

²⁴Also, note that dX_t and dY_t may be negative. The interpretation is that X_t and Y_t are expected total budgets available for the remainder of the campaign, where the expectation is formed at time t . Depending on the level of relative popularity, the candidates revise their expected future inflow of funds and adjust their spending choices accordingly.

Finally, let us assume for tractability that for the process (1) governing the evolution of popularity, we have $p(x, y) = \log(x/y)$.²⁵

Proposition 7. *In the model with evolving budgets, for every N, T , and $\lambda > 0$, there exists $-\eta < 0$ such that whenever $a - b \geq -\eta$, there is an essentially unique equilibrium. For all $t \in \mathcal{T}$, if $X_t, Y_t > 0$, then in equilibrium, $x_t/X_t = y_t/Y_t$.*

That the equal spending ratio result holds in this setting follows from the assumption that $p(x, y) = \log(x/y)$, which satisfies the ratio scale invariance condition. In fact, if this condition is satisfied with $p(x, y) = q(x/y)$ for some q , the equal spending ratio result would hold even if p does not take the log form so long as q is a strictly increasing, strictly concave function. However, for other specifications of q besides log, it is typically not possible to provide a closed form solution for the equilibrium spending decisions—see the proof of Proposition A.1 in Appendix A.5 for the closed form solution, which is necessarily stochastic and popularity-dependent (since popularity shocks affect the candidates’ budgets).

Proposition 7 states that the equal spending ratio result holds only if $a - b$ is not too negative. To understand this condition, note that when $a < 0 < b$, there is a negative feedback between popularity and the budget flow: a candidate’s budget increases when she is less popular than her opponent. The condition $a - b \geq -\eta$ puts a bound on how negative this feedback can be. If this condition is not satisfied, candidates may want to *reduce* their popularity as much as they can in the early stages of the campaign to accumulate a larger war chest that they can deploy in the later stages of the campaign. This could undermine the existence of an equilibrium in pure strategies.

How does the allocation of spending over time vary with the feedback parameters a and b that affect the budget processes? In the baseline model, when $\lambda > 0$ popularity leads decay. As a result, candidates’ spending is increasing over time. However, in this extension, if $b < 0 < a$ then there is a new force working in the opposite direction: spending to build early leads may be advantageous because it results in faster growth of the war chest. This mitigates the disincentive to spend early.

We establish this intuition formally. Recall that r_n defined for Section 2.3 gave the ratio of equilibrium spending in consecutive periods, n and $n + 1$. For this extension with evolving budgets, we define the analogous ratio, which we show in the appendix to be the same for both candidates:

$$\tilde{r}_n = \frac{x_{(n+1)\Delta}/X_{(n+1)\Delta}}{x_{n\Delta}/X_{n\Delta}} = \frac{y_{(n+1)\Delta}/Y_{(n+1)\Delta}}{y_{n\Delta}/Y_{n\Delta}}$$

We also show in the appendix that this ratio depends on the budget feedback parameters, a and b , only through the difference $a - b$.

²⁵This assumption guarantees that the law of motion of $(X_{n\Delta}, Y_{n\Delta}, Z_{n\Delta})$, is a function of candidates spending decisions only through their ratio. In this respect, it can be regarded as a strengthening of the ratio scale invariance assumption introduced to analyze the case in which money leftover is valuable.

Proposition 8. Fix the number of periods N , total time $T = N\Delta$, and consider the case in which $\lambda > 0$. Then, in equilibrium, for all n , if $a - b$ is sufficiently small then the consecutive period spending ratio \tilde{r}_n conditional on the history up to period n is (i) greater than 1, (ii) increasing in λ , and (iii) decreasing in $a - b$.

Our baseline model with the choice of $p(x, y) = \log(x/y)$ is a special case of this model with evolving budgets in which the total budget is constant over time: $a = b = \sigma_X = \sigma_Y = 0$. By Proposition 8, if we start from this special case and we increase the difference $a - b$ from zero, spending plans becomes more balanced over time: candidates spend more in earlier periods of the race.²⁶

Other Specifications. The popularity process can feed back into candidates' budgets also in other ways. For example, contributions may be higher when the race is close ($|z_t|$ is small), and lower when one of the candidates has a solid lead ($|z_t|$ is large). To capture this possibility, we can modify the budgets' laws of motion as follows:

$$\frac{dX_t}{X_t} = \frac{a}{1 + z_t^2} dt + \sigma_X dW_t^X \quad \text{and} \quad \frac{dY_t}{Y_t} = \frac{b}{1 + z_t^2} dt + \sigma_Y dW_t^Y,$$

with $a, b > 0$. Proposition A.2 in Appendix A.5 shows that the equal spending ratio result continues to hold in this case. However, in this setting closed-form characterizations of the spending path cannot be obtained in general. This is because the drifts of the budget processes depend non-linearly on popularity. Nevertheless, one special case in which closed-form solutions can be obtained occurs when $a = b$. Under this assumption, the percent change in campaign budgets arising from movements in relative popularity is the same for both candidates. As a result, the interior equilibrium is essentially unique and is the same as the one derived in Proposition A.1 of Appendix A.5, for the special case in which $a = b$.

²⁶The results of Proposition 8 do not necessarily hold when $a - b$ is very large. We have examples in which \tilde{r}_n is *increasing* in $a - b$ for large λ , n , and $a - b$. (One such example is $\lambda = 0.8$, $\Delta = 0.9$, and $n = a - b = 10$.) The intuition behind these examples rests on the fact that when the degree of mean reversion is high, then it is important for candidates to build up a large war chest that they can deploy in the final stages of the race. If the election date is distant and $a - b$ is large, then early spending is mostly for the purpose of building up these resources. But spending too much in any one period, especially an early period, is risky: if the resource stock does not grow (or even if it grows but insufficiently) then there is less money, and hence not much opportunity, to grow it further in the subsequent periods. Since p is concave, the candidates would like to have many attempts to grow the war chest early on, and this is even more the case as the importance of the relative feedback $a - b$ gets large.

4 A Model of the Electorate

4.1 Foundations for the Popularity Process

In the baseline model, we assumed that relative popularity evolves over time as the mean-reverting Brownian motion given by (1). We now provide a model of the electorate that justifies this assumption.

Voters are of three types: *core-1 voters*, who favor candidate 1, *core-2 voters*, who favor candidate 2, and *impressionable voters*, who are indifferent between the two candidates. The relative share of core-1 and core-2 voters in the population depends on a state of the world $\theta \in \{1, 2\}$. In state 1, the fraction of core-1 voters is $\bar{\nu}$ and the fraction of core-2 voters is $\underline{\nu}$. In state 2, the fraction of core-1 voters is $\underline{\nu}$, while the share of core-2 voters is $\bar{\nu}$. We assume that $\bar{\nu} > \underline{\nu}$ and that $\bar{\nu} + \underline{\nu} < 1$. The remaining $1 - \bar{\nu} - \underline{\nu}$ fraction of voters are impressionable voters. The state θ is realized at the beginning of the game and it is seen by the candidates.

Candidates compete as described in our baseline model. Thus, their choices are still given by $(x_{n\Delta})_{n=0}^{N-1}$ and $(y_{n\Delta})_{n=0}^{N-1}$, which we interpret as the amounts spent in campaign ads. To keep things simple, we assume that candidates cannot target specific groups of voters with their ads.

Core voters turn out to vote for their favored candidate if and only if they see that candidate's campaign ads: they become politically active if and only if they become politically informed through ads (e.g., they learn that the candidate's policy positions match their own political preferences). Otherwise, they do not vote. In particular, given profile of spending $(x_{m\Delta}, y_{m\Delta})_{m=0}^n$, candidate 1's ads mobilize a measure $Q_1(\theta, (x_{m\Delta})_{m=0}^n)$ of core-1 voters, while candidate 2's ads mobilize a measure $Q_2(\theta, (y_{m\Delta})_{m=0}^n)$ of core-2 voters. Let $\delta(\theta, (x_{m\Delta}, y_{m\Delta})_{m=0}^n) := Q_2(\theta, (y_{m\Delta})_{m=0}^n) - Q_1(\theta, (x_{m\Delta})_{m=0}^n)$. We assume that for all $(\theta, (x_{m\Delta}, y_{m\Delta})_{m=0}^n)$,

$$|\delta(\theta, (x_{m\Delta}, y_{m\Delta})_{m=0}^n)| < 1 - \bar{\nu} - \underline{\nu}.$$

This implies that if a candidate is supported by all impressionable voters, then she can win the election regardless of the state.

Impressionable voters are indifferent between the candidates and vote based on their relative goodwill towards the candidates. This goodwill evolves throughout the campaign. Goodwill is common among impressionable voters and depends on the amount spent by candidates. In particular, if the stock of relative goodwill of candidate 1 to candidate 2 at time $t = n\Delta$ is $\tilde{Z}_{n\Delta}$, the next-period stock of relative goodwill, $\tilde{Z}_{(n+1)\Delta}$, is stochastic and given by:

$$\tilde{Z}_{(n+1)\Delta} = (\tilde{Z}_{n\Delta})^{\exp(-\lambda\Delta)} (\tilde{p}(x_{n\Delta}, y_{n\Delta}))^{1-\exp(-\lambda\Delta)} \tilde{\varepsilon}_{n\Delta}, \quad (10)$$

where $\tilde{\varepsilon}_{n\Delta}$ follows a log-normal distribution, $\lambda\Delta > 0$, and \tilde{p} is increasing in the first argument and decreasing in the second. Thus, goodwill in the next period is "produced" by a Cobb-Douglas function: it depreciates because voters "forget" some of the goodwill

previously accumulated—the $(\tilde{Z}_{n\Delta})^{\exp(-\lambda\Delta)}$ part—but additional goodwill is built through the ads—the $\tilde{p}(x_{n\Delta}, y_{n\Delta})^{1-\exp(-\lambda\Delta)}$ part.²⁷ If on the election day, $T = N\Delta$, the stock of goodwill is \tilde{Z}_T , the mass of impressionable voters who vote for candidate 1 minus the mass that votes for candidate 2 is given by $H(\tilde{Z}_T)$, where H is a strictly increasing function ranging from $-(1 - \bar{\nu} - \underline{\nu})$ to $+(1 - \bar{\nu} - \underline{\nu})$.

Putting together core and impressionable voters, we can conclude that candidate 1 wins the election if and only if $H(\tilde{Z}_T) - \delta(\theta, (x_{m\Delta}, y_{m\Delta})_{m=0}^{N-1}) \geq 0$, or if

$$\tilde{Z}_T \geq H^{-1} \left(\delta(\theta, (x_{m\Delta}, y_{m\Delta})_{m=0}^{N-1}) \right). \quad (11)$$

Define $Z_{n\Delta} = \log \tilde{Z}_{n\Delta}$ and $p(x_{n\Delta}, y_{n\Delta}) = (1 - e^{-\lambda\Delta}) \log \tilde{p}(x_{n\Delta}, y_{n\Delta})$ and $\varepsilon_{n\Delta} = \log \tilde{\varepsilon}_{n\Delta}$. From (10), $Z_{n\Delta}$ evolves according to $Z_{(n+1)\Delta} = e^{-\lambda\Delta} Z_{n\Delta} + p(x_{n\Delta}, y_{n\Delta}) + \varepsilon_{n\Delta}$ and the probability of candidate 1 winning is (we suppress the argument of δ to simplify notation):

$$\Pr \left[\tilde{Z}_T \geq H^{-1}(\delta) \right] = 1 - \Phi \left(H^{-1}(\delta) - \sum_{m=0}^{N-1} \left[e^{-\lambda\Delta(N-m-1)} p(x_{m\Delta}, y_{m\Delta}) + \varepsilon_{m\Delta} \right] \right)$$

Regardless of the actual $(\varepsilon_{m\Delta})_{m=0}^{N-1}$, the above expression is increasing in the term $-H^{-1}(\delta) + \sum_{m=0}^{N-1} e^{-\lambda\Delta(N-m-1)} p(x_{m\Delta}, y_{m\Delta})$. Then, we can assume that candidate 1 maximizes this expression, while candidate 2 minimizes it (see Appendix B.1 for details).

If $Q_1(\theta, (x_{m\Delta})_{m=0}^{N-1})$ and $Q_2(\theta, (y_{m\Delta})_{m=0}^{N-1})$ are linear in $(x_{m\Delta})_{m=0}^{N-1}$ and $(y_{m\Delta})_{m=0}^{N-1}$ then we abuse our notation and rewrite them as $Q_1(\theta, \sum_{n=0}^{N-1} x_{n\Delta})$ and $Q_2(\theta, \sum_{m=0}^{N-1} y_{m\Delta})$. This case is relevant if ads by candidate i have the same probability of informing a core- i voter in any period. Since money left over has no value in the baseline model, we have $\sum_{n=0}^{N-1} x_{n\Delta} = X_0$ and $\sum_{n=0}^{N-1} y_{n\Delta} = Y_0$. So candidate 1 maximizes $\sum_{n=0}^{N-1} e^{-\lambda\Delta(N-m-1)} p(x_{n\Delta}, y_{n\Delta})$, while candidate 2 minimizes it, as with equation (4) of the baseline model.

Proposition 9. *If Q_1 and Q_2 are linear in spending, then the equilibrium of the model in this section is outcome-equivalent to the equilibrium of the baseline model.*

4.2 Welfare and Policy Implications

Besides providing a foundation for the law of motion of the popularity process assumed in the baseline model, the analysis in this section can also be used to study welfare and policy interventions. Suppose that the goal of society is to elect candidate $i = \theta$ in state θ .²⁸ Further, assume that by the end of the campaign, if the state is θ , spending is effective

²⁷This setting is inspired by the stochastic goodwill models studied in the marketing literature (see, e.g., Nerlove and Arrow, 1962, Marinelli, 2007). The assumption of impressionable voters also draws on recent work in political economy by Andonie and Diermeier (2019).

²⁸Because in state θ the majority of core voters prefers the policy proposed by candidate θ , while impressionable voters are indifferent between the two policies, this welfare criterion can be justified assuming a majoritarian social welfare function over policies.

enough so that the majority of core voters support candidate θ .²⁹ This says that for all spending paths $(x_{m\Delta}, y_{m\Delta})_{m=0}^{N-1}$:

$$H^{-1}\left(\delta(1, (x_{m\Delta}, y_{m\Delta})_{m=0}^{N-1})\right) < 0 < H^{-1}\left(\delta(2, (x_{m\Delta}, y_{m\Delta})_{m=0}^{N-1})\right) \quad (12)$$

Impressionable voters introduce noise in the electoral outcome. Due to this noise, the welfare-inferior candidate may win. In addition, if Q_1 and Q_2 are linear, candidates increase their spending over time exactly because of impressionable voters.³⁰ Hence, a policy implication of our model is that spending caps that limit campaign spending in later periods are welfare-enhancing. Since the support among core voters is linear in total spending, our model suggests a particularly stark policy intervention: to ban all spending from period 2 onwards. This intervention has no effect on core voters, but it reduces the impact of impressionable voters on the electoral outcome.

The starkness of this policy recommendation hinges on the linearity assumption concerning the effect of spending among core voters. Nonetheless, the qualitative insight is robust to other specifications: since impressionable voters introduce noise in the electoral outcome, policies that induce a marginal reallocation of resources towards earlier periods help elect candidate θ in state θ and they are thus welfare improving.

To see this, suppose that the marginal effect of campaigning among core voters is decreasing in the amount spent in every period—e.g., as a candidate increases ad spending, she is increasingly more likely to reach voters that are already informed about her policy. Let $H^{-1}(\delta) = c\delta$ with $c > 0$, $p(x_{n\Delta}, y_{n\Delta}) = x_{n\Delta}^\beta - y_{n\Delta}^\beta$ with $\beta \in (0, 1)$,

$$Q_1\left(\theta, (x_{n\Delta})_{n=0}^{N-1}\right) = c_{1,\theta} \sum_{n=0}^{N-1} x_{n\Delta}^\beta \quad \text{and} \quad Q_2\left(\theta, (y_{n\Delta})_{n=0}^{N-1}\right) = c_{2,\theta} \sum_{n=0}^{N-1} y_{n\Delta}^\beta$$

where $c_{1,1}, c_{2,2} > c_{1,2}, c_{2,1} > 0$. Thus, in state i , candidate i 's ads inform more core voters than the ads of the opponent.

Given this specification, the first order conditions of the two candidates at any period $n \in \{0, \Delta, \dots, (N-2)\Delta\}$ are

$$c_{1,\theta}\beta x_{n\Delta}^{\beta-1} + \beta e^{-\lambda\Delta(N-n-1)}x_{n\Delta}^{\beta-1} = c_{1,\theta}\beta x_{(n+1)\Delta}^{\beta-1} + \beta e^{-\lambda\Delta(N-n-2)}x_{(n+1)\Delta}^{\beta-1} \quad (13)$$

$$c_{2,\theta}\beta y_{n\Delta}^{\beta-1} + \beta e^{-\lambda\Delta(N-n-1)}y_{n\Delta}^{\beta-1} = c_{2,\theta}\beta y_{(n+1)\Delta}^{\beta-1} + \beta e^{-\lambda\Delta(N-n-2)}y_{(n+1)\Delta}^{\beta-1} \quad (14)$$

²⁹If this were not the case, a spending limit would still be welfare improving, but only because it would level the playing field among the two candidates.

³⁰Impressionable voters forget, while core voters do not. Furthermore, the return to advertising among core voters is linear in the total amount spent and does not depend on the timing of spending. Without impressionable voters, candidates would thus have no incentive to increase their spending over time.

We can derive the growth rates of $x_{n\Delta}$ and $y_{n\Delta}$ at a given period n from the above equations. Defining $r_{1,\theta,n} := x_{(n+1)\Delta}/x_{n\Delta}$ and $r_{2,\theta,n} := y_{(n+1)\Delta}/y_{n\Delta}$ for each n , we see that

$$r_{i,\theta,n} = \left(\frac{c_{i,\theta} + e^{-\lambda\Delta(N-n-1)}}{c_{i,\theta} + e^{-\lambda\Delta(N-n-2)}} \right)^{\frac{1}{1-\beta}}, \quad i = 1, 2.$$

Therefore, because $c_{i,i} > c_{j,i}$, the candidate who spends money more smoothly in state θ is candidate θ . In other words, for any n , $r_{1,\theta,n} < r_{2,\theta,n}$ if $\theta = 1$ and $r_{1,\theta,n} > r_{2,\theta,n}$ if $\theta = 2$. To understand why, recall that the main reason that candidates spend money in the early stages of the campaign is to inform core voters. Because the candidate i with a higher $c_{i,\theta}$ is able to reach more core voters, she is also the candidate with the greater incentive to spend resources in the early stages of the campaign.

Regulations that incentivize smoother spending patterns are thus beneficial for the candidate with more core voters in state θ . To be more concrete, consider a policy that bounds candidate i 's growth rate of spending to the rate that candidate i would choose in state $\theta = i$. Under this policy, candidate i is elected more often than in the unregulated equilibrium if and only if $\theta = i$. Indeed, given the zero-sum payoff structure of the game, a policy intervention that does not affect candidate i 's spending in state $\theta = i$, but constrains it in state $\theta \neq i$ is beneficial for candidate i if and only if the state is $\theta = i$. Any such policy intervention will be welfare-improving, given our assumptions.

However, implementing such a policy is difficult since it depends on features of the election that may not be observable to policymakers. An alternative policy that we consider is that that of “campaign silence” which prohibits all campaign spending in the final stages of a campaign. Such regulations already exist in at least 40 countries, where electoral campaigning, polling, and/or certain kinds of press coverage of elections are banned for periods ranging from two days to two weeks before an election.³¹ Aspects of these policies are of course motivated by other considerations, but our model provides a new rationale for how time-dependent regulation can be welfare-improving. We provide a quantitative analysis of the welfare gains to campaign silence in Appendix C. The analysis shows how a policy that prohibits spending in the final week of an election improves welfare upon the laissez-faire benchmark, for high values of λ . It also shows how these time-dependent regulations outperform a static aggregate spending cap like the one that has been enacted in several electoral campaigns. These include the U.K. races studied by Fourinaies (2018), as well as the Brazilian mayoral races studied by Avis et al. (2017).³²

³¹See ACE Electoral Knowledge Network: <http://aceproject.org/epic-en/CDMap?question=ME062>.

³²Restrictions on campaign spending were deemed unconstitutional in the U.S. by the landmark Supreme Court decision in *Buckley v. Valeo* 1976. In this context, our model points to the welfare costs arising from the constitutional commitment to unregulated spending in elections.

5 Campaign Spending in Practice

We now offer a closer look at actual electoral spending data through the lens of our model. The main question we ask is: What decay rates (or, equivalently, what values of λ) are consistent with actual patterns of campaign spending? We recover estimates of the perceived decay rate assuming the specification of our model in Section 2.3 for $\beta = -1$.

5.1 Data

We focus on subnational American elections, namely U.S. House, Senate, and gubernatorial elections in the period 2000 to 2014.

Spending in our model refers to all spending—TV ads, calls, mailers, door-to-door canvassing visits—that directly affects the candidates’ relative popularity. But for some of these categories of spending, it is not straightforward to separate out the part of spending that has a direct impact on relative popularity from the part that does not (e.g. fixed administrative costs). For one category, namely TV advertising, it is straightforward to do this, so we focus exclusively on TV ad spending. Television advertising constitutes around 35% of the total expenditures by congressional candidates, and approximately 90% of all advertising expenditure during the period studied (Albert, 2017). We proceed under the assumption that any residual spending on other types of campaign activities that directly affect relative popularity is proportional to spending on TV ads.

Our TV ads spending data are from the Wesleyan Media Project and the Wisconsin Advertising Database. For each election in which TV ads were bought, the database contains information about the candidate that each ad supports, the date it was aired, and the estimated cost. For the year 2000, the data covers only the 75 largest Designated Market Areas (DMAs), and for years 2002-2004, it covers only the 100 largest DMAs. The data from 2006 onwards covers all of the 210 DMAs. We obtain the amount spent on ads from total ads bought and the price per ad. For 2006, where ad price data are missing, we estimate prices using ad prices in 2008.³³

We focus on races where the leading two candidates in terms of vote share are from the Democratic and the Republican party. We label the Democratic candidate as candidate 1 and the Republican candidate as candidate 2, so that x_t, X_0 , etc. refer to the Democrat’s spending, budget, etc. and y_t, Y_0 , etc. refer to the Republican’s.

We aggregate ad spending made on behalf of the two major parties’ candidates by week and focus on the twelve weeks leading to election day, though we will drop the final week

³³One concern with this approach could be that if prices increase as the election approaches, then the increase in total spending over time confounds the price increase with increased advertising. However, federal regulations limit the ability of TV stations to increase ad prices as the election approaches, and instead requires them to charge political candidates “the lowest unit charge of the station for the same class and amount of time for the same period” (Chapter 5 of Title 47 of the United States Code 315, Subchapter III, Part 1, Section 315, 1934). This fact allays some of this concern.

Table 1: Descriptive Statistics

	N	Open Seat Election	Incumbent Competing	No Excuse Early Voting	Average total spending	Average spending difference
Senate	122	68	54	82	6019 (5627)	1962 (2921)
Governor	133	59	74	92	5980 (9254)	3173 (6, 337)
House	346	97	249	223	1533 (1304)	521 (615)
Overall	601	224	377	397	3428 (5581)	1401 (3, 461)

Week	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
<i>Senate</i>											
Avg. spending	196 (291)	250 (328)	266 (403)	314 (487)	357 (401)	477 (505)	545 (577)	652 (724)	716 (803)	860 (947)	1,002 (1,047)
Share spending 0	0.270	0.180	0.123	0.082	0.008	0	0	0	0	0	0
<i>Governor</i>											
Avg. spending	262 (632)	253 (468)	258 (424)	316 (581)	420 (865)	416 (579)	530 (1,249)	597 (1,015)	701 (1,305)	800 (1,523)	1,019 (1,956)
Share spending 0	0.297	0.207	0.139	0.068	0.030	0	0	0	0	0	0
<i>House</i>											
Avg. spending	17 (41)	27 (55)	38 (57)	56 (85)	83 (93)	120 (134)	137 (134)	177 (182)	212 (219)	250 (270)	303 (340)
Share spending 0	0.653	0.545	0.386	0.246	0.095	0	0	0	0	0	0

Note: The upper panel reports the breakdown of elections that are open seat versus those that have an incumbent running, the numbers in which voters can vote early without an excuse to do so, average spending levels by the candidates, and the average difference in spending between the two candidates, all by election type. The lower panel presents average spending for each week in our dataset, and the percent of candidates spending 0 in each week, by election type. Standard deviations are in parentheses. All monetary amounts are in units of \$1,000.

which is typically incomplete since elections are held on Tuesdays. For our baseline analysis, we exclude elections that are clearly not genuine contests to which our model does not apply, defining these to be elections in which one of the candidates did not spend anything for more than half of the period studied. This leaves us with 346 House, 122 Senate, and 133 gubernatorial elections.³⁴ We focus on the last twelve weeks mainly because we want to restrict attention to the general election campaign. We define the total budgets of the candidates to be the total amount that they spend over these twelve weeks.³⁵ However, in the appendix, we also include the replication of our empirical analyses with a larger dataset excluding fewer elections (leaving us with about 1100 elections over 14 years), and a longer time period (20 weeks instead of 12 weeks).

³⁴A tabulation of these elections is given in the Appendix.

³⁵In some elections, the primaries are held closer than twelve weeks to the general election date, but ad spending for the general election in the period prior to the primary elections is typically zero. In the rare cases where ad spending for primary elections happens, we exclude it and focus only on spending for the general election.

Summary statistics for our baseline sample are given in Table 1. There is considerable difference in the amount of spending between state-wide and House elections, with another key difference being the time at which candidates start spending positive amounts. For statewide races, candidates spend about \$6 million on TV ads on average, with most candidates starting spending 12 weeks prior to the election. For House races, they spend \$1.5 million on average and the majority of candidates start spending 9 weeks before the election. Because of these differences, we will include these disaggregations in the models that we use to estimate perceived decay rates.

Model Diagnostics. The main robust prediction of our set of models is the equal spending ratio result. Figure 1 in the introduction reveals that there are some violations of this prediction in the data, which are in part due to noisy observations of the actual spending path, but may also be driven by factors not covered by our models. In Appendix D we look at the extent to which the equal spending ratio result holds in the data. Looking at twelve weeks of data, we find that the candidates’ weekly spending ratios are within 10 percentage points of each others’ in 85% of the dataset, and within 5 percentage points of each others’ in 65% of the data. Even in the final six weeks of the campaign where candidates spend larger amounts, they are within 10 percentage points of each others’ in 75% of the dataset, and within 5 percentage points of each others’ in about half. Overall, the equal spending ratio result holds up surprisingly well in the data.

Another prediction of the model in Section 2.3 that we estimate is the constant spending growth result, $r_t = r$ for all $t \in \mathcal{T}$. We also examine this prediction in Appendix D. Overall, we find somewhat less robust support for this prediction. Candidates remain within half a standard deviation of their mean consecutive period spending ratios for 5.25 out of the 10 weeks that we can calculate this ratio. On an average week, 54% of candidates are within this range. However, unlike the equal spending ratio result, the constant spending growth prediction is violated to a smaller extent as the election approaches and candidates begin to spend more substantial amounts. We show that violations of this prediction are not explained by races with early voting, those that are close in terms of spending or final vote share, or those in which an incumbent is running for re-election.

However, one possibility that we cannot rule out is that violations of the constant spending growth prediction are due to candidates not being able to forecast how much money they will have by the end of the race. Given that the extension on evolving budgets in Section 3.4 does not predict constant spending growth, we estimate perceived decay rates for various specifications of this extension as well.

5.2 Perceived Decay Rates

We estimate one of the key parameters of our model, the speed of reversion λ of the popularity process using spending data. For the estimation we assume that

$$p(x, y) = \alpha_1 \log x - \alpha_2 \log y. \quad (15)$$

Under this assumption, the consecutive period spending ratio is constant and equal to $r = e^{\lambda\Delta}$ (see Section 2.3). We make two observations before proceeding. First, parameters $\alpha_1, \alpha_2 > 0$ are not identified from spending data alone; but they are irrelevant for the estimation of decay rates, so we do not need to specify them. Second, λ and Δ cannot be separately identified from the spending data; only their product $\lambda\Delta$ is identified. So we can choose Δ arbitrarily. The next proposition states this result.

Proposition 10. *Let Γ^Δ denote the game of our baseline model, and consider a modified game $\Gamma^{\bar{\Delta}}$ in which all other parameters are the same but the candidates take actions more frequently at time periods of length $\bar{\Delta} = \Delta/M$, where M is a positive integer. Let x_t^Δ and y_t^Δ be the equilibrium amounts that the candidates spend in game Γ^Δ at times $t \in \mathcal{T} := \{0, \Delta, \dots, (N-1)\Delta\}$ and $x_t^{\bar{\Delta}}$ and $y_t^{\bar{\Delta}}$ be the equilibrium amounts that they spend in game $\Gamma^{\bar{\Delta}}$ at times $t \in \bar{\mathcal{T}} := \{0, \bar{\Delta}, \dots, (N-1)\Delta + (M-1)\bar{\Delta}\}$. Then, for all times $t \in \mathcal{T}$,*

$$x_t^\Delta = \sum_{m=0}^{M-1} x_{t+m\bar{\Delta}}^{\bar{\Delta}} \quad \text{and} \quad y_t^\Delta = \sum_{m=0}^{M-1} y_{t+m\bar{\Delta}}^{\bar{\Delta}}$$

The result implies, for example, that if the candidates are making spending decisions every day, but we collect spending data every week—so that $M = 7$ —, then our estimate of λ from weekly data would be $1/7^{\text{th}}$ of the λ for daily data. Therefore, for our analysis of spending in the final twelve weeks of each election, we fix $\Delta = 1$ week, set $T = 11$ (recall that we drop the final incomplete week), and estimate λ for these values of Δ and T . In fact, we report estimates of the weekly *decay rate* in polling leads, $1 - e^{-\lambda\Delta}$, which is one-to-one with λ and has a clear interpretation.

We estimate election-specific perceived λ from spending data using a hierarchical Bayes model. Let λ_j denote the perceived speed of reversion in election $j = 1, \dots, J$. Given that we observe zero spending by some candidates in the early weeks, and $x_{t,j}/X_{t,j}, y_{t,j}/Y_{t,j} \in [0, 1]$, we model this data generating process using a zero-inflated truncated normal distribution with mean

$$\mu(\lambda_j, t) = \frac{e^{-\lambda_j(T-t-\Delta)} - e^{-\lambda_j(T-t)}}{1 - e^{-\lambda_j(T-t)}} \quad (16)$$

This expression is the share of initial budget spent by each candidate in the baseline model when $p(x, y)$ is given by (15). We specify the following statistical model for the likelihood

of observed spending and for the priors of the underlying parameters:

$$\begin{aligned}
\frac{x_{t,j}}{X_{t,j}}, \frac{y_{t,j}}{Y_{t,j}} &\sim (1 - \varsigma_t) \times \text{Truncated } [0,1] \text{ Normal } (\mu(\lambda_j, t), \sigma_{\text{spend}}^2) \\
\varsigma_t &\sim \text{Bernoulli}((T - t)^2 \varrho_{\text{type}}), \quad \text{type} = \text{S, H, G} \\
\varrho_{\text{S}}, \varrho_{\text{H}}, \varrho_{\text{G}} &\sim \text{Uniform}(0, 0.00826) \\
\lambda_j &\sim \text{logNormal}(\beta_{\text{O}} \text{Open}_j + \beta_{\text{E}} \text{Early}_j + \beta_{\text{S}} \text{Senate}_j + \beta_{\text{H}} \text{House}_j + \beta_{\text{G}} \text{Gov}_j, \sigma_\lambda^2) \\
\beta_{\text{O}}, \beta_{\text{E}}, \beta_{\text{S}}, \beta_{\text{H}}, \beta_{\text{G}} &\sim \text{Normal}(0, 1) \\
\sigma_\lambda^2 &\sim \text{Scale-inv-}\chi^2(5, 1) \\
\sigma_{\text{spend}}^2 &\sim \text{Scale-inv-}\chi^2(1, 10)
\end{aligned}$$

Notation is as follows. ς_t is a Bernoulli random variable for observing zero spending at time t . Its mean decreases over time as election day approaches.³⁶ ϱ_{type} is the baseline probability of observing zero spending during election week $t = 11$, which we allow to be different across election types (S for Senate, H for House, and G for governor races), motivated by our observations in Table 1. Since $T - t$ can take a maximum value of 11, ϱ_{type} can take a maximum value of 0.00826. We posit a uniform prior over $[0, 0.00826]$ for these baseline probabilities. The speed of reversion λ_j has a log-normal distribution whose parameters depend on election characteristics: **Open** indicates an open-seat election, **Early** indicates early voting was available, and the **Senate**, **House** and **Gov** variables here are also indicators for election type. The β variables are the corresponding coefficients. The interpretation of our theoretical model (see the remark at the end of Section 2) allows λ to vary depending on election characteristics, and the inclusion of these variables as potential determinants of the decay rate are driven by what we see in Table D.1. Finally, σ_λ^2 and σ_{spend}^2 are common variance parameters for, respectively, the election specific λ_j , and the observed spending ratios $x_{t,j}/X_{t,j}$ and $y_{t,j}/Y_{t,j}$, with scaled inverse chi-square priors (a common choice for variance parameters). The scales are chosen appropriately for these variables. See Figure 3 for a representation of this model.³⁷

We use No-U-Turn sampling, a Hamiltonian Monte Carlo based method, to get our posteriors for the parameters (see Hoffman and Gelman, 2014). We report the Bayesian credible intervals from the posterior distributions for the key model parameters in Table 2. As the table shows, we do not find meaningful differences in the estimates between open

³⁶Thus, we model the truncation process with a time-dependent parameter. Decoupling the truncation from the spending process allows us to estimate the underlying decay rate parameters, without our choice of time horizon affecting them mechanically.

³⁷Under mild regularity conditions, the posterior Bayesian credible intervals from this hierarchical Bayes model asymptotically approach the confidence intervals of the same parameters obtained by standard frequentist approaches using any efficient estimation method. This is the Bernstein-von Mises Theorem; see, for example, Section 10.2 of Van der Vaart (2000). Given the complexity of an equivalent frequentist approach to get estimates of all the listed parameters, we take the more transparent Bayesian approach.

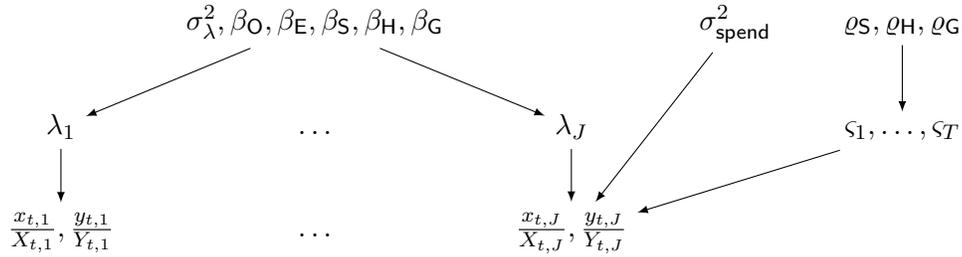


Figure 3: The picture above depicts the hierarchical structure of our model. At the highest level, are the coefficients for election characteristics (open seat, early voting, and election type), the common variance parameters for the speed of reversion and for weekly spending, and the baseline zero-spending probabilities. These determine the distributions of election specific mean reversion parameters, and weekly zero-spending probabilities in the middle layer. Finally, all of these parameters determine the distribution of weekly spending at the lowest level.

seat elections and elections in which an incumbent competes, or between elections with early voting and elections without. The differences across election types are also minimal. Although House candidates start spending on ads later, the estimate for this type of election is similar to those of state-wide elections. The lack of a statistically meaningful difference in the estimates between statewide elections and congressional district elections is a notable qualitative result.

We also replicate our analysis using 20 weeks of spending data and a larger dataset of 1163 elections in which we keep all contests where two candidates spend non-zero amounts for at least two weeks. Overall, our estimates are robust to the choice of time horizon and to throwing away fewer elections. These results are reported in Appendix E.1.

We then transform the posteriors of election-specific λ_j to posteriors on perceived weekly decay rates—recall that the weekly decay rate is $1 - e^{-\lambda_j}$). These estimates complement existing research on the estimates of actual decay rates using survey and experimental data. For example, using survey data and an exponential decay model similar to ours, Hill et al. (2013) recovers an average daily decay rate in the persuasive effects of political advertising of 52.4% in 2006 U.S. elections. This corresponds to a 99% weekly decay, though their 95% confidence interval for this estimate covers the $[0, 100\%]$ interval. Similarly, using a field-experimental approach, Gerber et al. (2011) recovers a weekly decay rate of 88%, though in their case, the estimates vary substantially according to the specification of their model. For example, their 3rd order polynomial distributed lag model estimates show that the standing of the advertising candidate increases by 4.07 percentage points in the week that the ad is aired, and the effect goes down to 3.05 percentage points the following week (a 25% decay). In another specification, the first week effect is 6.48%, and goes down to 0.44% in the second (a 94% decay).³⁸

Parameter	\hat{R}	n_{eff}	mean	s.d.	2.5%	50%	97.5%
β_{O}	1.001	5491	-0.017	0.095	-0.203	-0.017	0.172
β_{E}	1.002	5972	-0.175	0.092	-0.357	-0.175	0.007
β_{S}	1.003	4418	-2.427	0.126	-2.681	-2.425	-2.189
β_{H}	1.003	3537	-2.552	0.090	-2.733	-2.551	-2.379
β_{G}	1.001	4290	-2.470	0.122	-2.712	-2.468	-2.238
ϱ_{S}	1.000	26717	0.003	0.002	0.000	0.003	0.007
ϱ_{H}	1.000	25982	0.003	0.000	0.003	0.003	0.003
ϱ_{G}	1.000	29000	0.003	0.002	0.000	0.003	0.007
σ_{spend}^2	1.000	15799	0.012	0.000	0.012	0.012	0.012
σ_{λ}^2	1.004	1698	0.378	0.060	0.272	0.374	0.504

Table 2: Model parameters with convergence diagnostics and 95 % Bayesian credible intervals for the 601 elections in our sample, and 12 weeks of data.

If we take the point estimates from these prior studies at face value, Figure 4 shows that the perceived decay rates of the candidates are considerably lower than previous estimates of actual decay rates. Our parameterized baseline model therefore suggests that candidates spend more in earlier weeks compared to what the decay rates estimated from the past literature would imply. On the other hand, since our estimates of the perceived decay rates appear to be within the large margins of error of prior estimates of actual decay rates, we can make no conclusive inferences on this.

That being said, it is also important to note that we estimate decay rates under the baseline model in which spending growth is constant. Although there is some support for this prediction, there are many election-weeks where the constant spending growth does not hold (see Appendix D.2). We postulate that one possible reason for these violations is that budget processes can be endogenous, as modeled in Section 3.4. In light of this, in Appendix E.2 we estimate decay rates using this extension as our underlying model. Since the parameter $a - b$ governing how leads in popularity translate into changes in budgets cannot be separately identified from the decay rate using spending data alone (and we do not have access to sufficiently rich data on the timing of campaign contributions), we do not set a prior for $a - b$ in the estimation. Instead, we use a grid of different values for $a - b$ and show that estimates of the perceived decay rate increase as $a - b$ increases. This extension thus provides another explanation for the apparent skew of candidates' spending toward the early weeks. We find that the observed pattern of spending by candidates can be consistent with the decay rates obtained from the experimental literature if there are budget feedback effects by which a popularity lead increases the subsequent week budget. For example, if we set $a - b = 0.5$, the median weekly decay rate is approximately 75%.

Finally, the distribution of decay rates suggests a considerable degree of heterogeneity between elections. This heterogeneity cannot be fully explained by election type, incumbency, or early voting status. This finding deserves further attention by future work.

³⁸The volatility of these estimates may be due to data limitation; see, e.g., Lewis and Rao (2015).

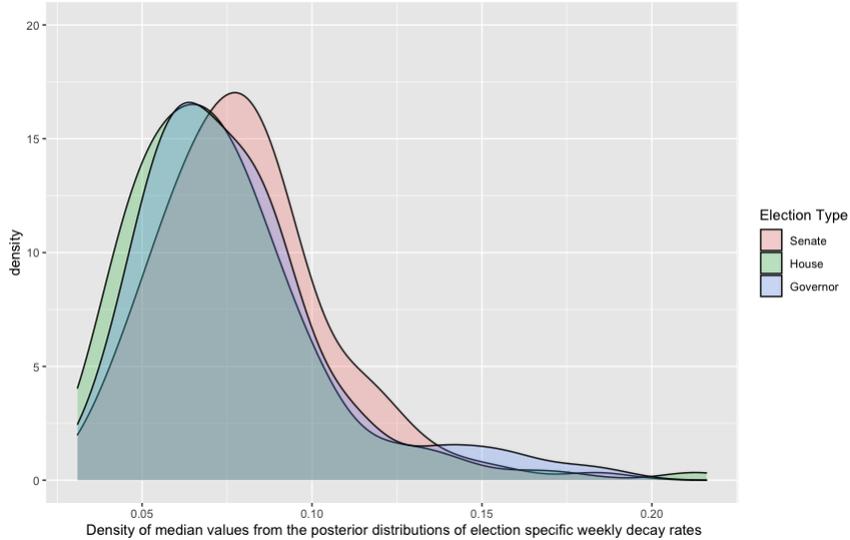


Figure 4: The distributions of median election specific perceived weekly decay rates by race type, estimated using 12 weeks of spending data.

5.3 Application: Campaign Silence and the Win Rates of Incumbents

In this section, we demonstrate the applicability of our decay rate estimates by studying the implications of the campaign silence policies mentioned in Section 4.2 for the win rates of incumbents. We continue to focus on twelve weeks of campaigning, setting $\Delta = 1$ week, and using estimates of λ_j for each election j from the previous section, along with the data on spending levels $x_{t,j}$ for the incumbent, and $y_{t,j}$ for the challenger. Thus we denote candidate 1 as the incumbent and candidate 2 as the challenger.

For this exercise, we require estimates of additional parameters, namely α_1, α_2 from equation (15), as well as the mean initial relative popularity for incumbents, z_0 , and the volatility of the Wiener process from equation (1), σ , or equivalently the variance of the errors in equation (3), which we denote σ_ε^2 . We normalize $\alpha_2 = 1$ and estimate the parameters α_1, z_0 and σ_ε^2 that we assume to be common across elections.

To estimate these parameters, we augment our existing data set on TV ad spending with polling data obtained from FiveThirtyEight, as well as final vote shares available from public records. Out of the 377 elections in our data set in which incumbents competed, we are able to find polling data for 268 of them, going back 3 weeks from election day on average. For these 268 elections, we average the results of polls conducted in the earliest available week, and define this week as time \hat{t}_j . These averages give us an estimate of the time \hat{t}_j value of the realized relative popularity $Z_{\hat{t}_j,j}$, which we denote $z_{\hat{t}_j,j}$. The final

realized vote shares are denoted $z_{T,j}$. Our popularity process implies that for election j

$$\left(\frac{z_{T,j} - (1 - e^{-\lambda_j}) \sum_{t=\hat{t}_j}^{11} e^{-\lambda_j(11-t)} (\alpha_1 \log x_{t,j} - \log y_{t,j})}{e^{-(11-\hat{t}_j)\lambda_j}} \right) \sim \mathcal{N} \left(z_{\hat{t}_j,j}, \left(\frac{\sum_{t=\hat{t}_j}^{11} e^{-2\lambda_j(11-t)}}{e^{-2(11-\hat{t}_j)\lambda_j}} \right) \right).$$

which comes from equation (4). We estimate α_1 by maximizing the likelihood function given the data and the estimates of λ_j . We estimate α_1 to be 0.982 with bootstrapped 95% confidence interval [0.968, 0.995]. Therefore, we find that incumbent spending is less effective than challenger spending, but not substantially so: the popularity process appears to be largely symmetric between incumbents and challengers.

Taking these estimates of α_1 , we turn to estimating the average lead of the incumbents at the start of the electoral campaign, z_0 . For election j , our model specifies that final popularity can be written as a function of initial popularity, the spending paths, λ_j , and α_1 , specified by the following relationship, similar to the one above:

$$\left(\frac{z_{T,j} - (1 - e^{-\lambda_j}) \sum_{t=0}^{11} e^{-\lambda_j(11-t)} (\alpha_1 \log x_{t,j} - \log y_{t,j})}{e^{-12\lambda_j}} \right) \sim \mathcal{N} \left(z_0, \left(\frac{\sum_{t=0}^{11} e^{-2\lambda_j(11-t)}}{e^{-24\lambda_j}} \right) \right).$$

Note that we assume that each election has its own realized value of initial relative popularity $z_{0,j}$ but that these are draws from a distribution with common mean z_0 .) We estimate z_0 from the empirical mean of the above transformed values. The estimated value for z_0 is 0.128, with bootstrapped 95% confidence interval (0.0, 0.257). Thus, we estimate that incumbents enjoy on average a 12.8% lead in relative popularity at the start of the campaign.

Finally, to calculate the probability of the incumbents winning an election, we estimate σ_ε^2 as follows. We estimate this as an average, using the fact that

$$v_j := \left(\frac{z_{T,j} - (1 - e^{-\lambda_j}) \sum_{t=0}^{11} e^{-\lambda_j(11-t)} (\alpha_1 \log x_{t,j} - \log y_{t,j}) + z_0 e^{-12\lambda_j}}{\left(\sum_{t=0}^{11} e^{-2\lambda_j(11-t)} \right)^{1/2}} \right) \sim \mathcal{N}(0, \sigma_\varepsilon^2)$$

and the approximation

$$\mathbb{E} \left(\left(\frac{\sum_{j=1}^J v_j}{J-1} \right)^{1/2} \right) \approx \sigma_\varepsilon$$

given to us by Cochran's Theorem, where J is the number of elections. The approximation holds for large values of J . We estimate σ_ε^2 to be 0.042.

Given the previous estimates, we can calculate the win probability for incumbents in all 377 elections under different regulatory settings and under the assumption that candidates play the equilibrium of our model. The results are depicted in Figure 5. Overall, we see that campaign silence can either help or hurt incumbents, and that it makes electoral contests

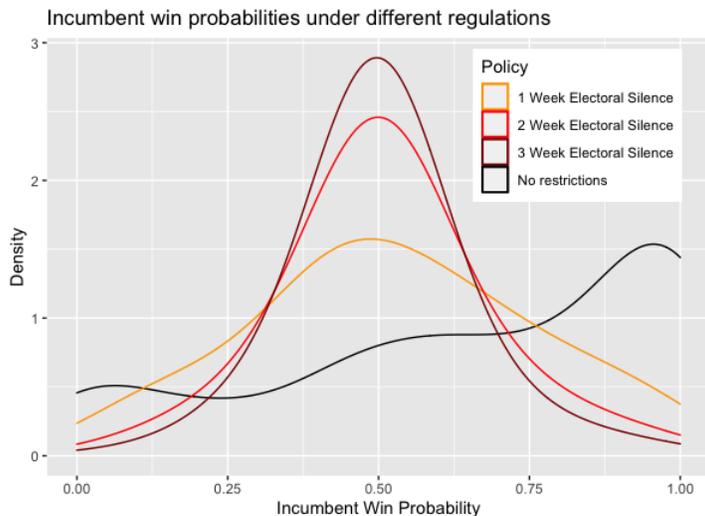


Figure 5: Density of the incumbent’s win probability under various regulations, calculated using the 377 elections in the data set in which an incumbent competes.

less lopsided. We find that even a single week of campaign silence moves the distribution of win probabilities of incumbents significantly closer to 50%, and makes the density of incumbent win probabilities much more symmetric around even.

6 Conclusion

We developed a model of electoral campaigns as dynamic contests, to study the optimal allocation of campaign resources over time when popularity leads tend to decay. We provide a tractable framework to analyze the dynamics of campaign spending and we identify conditions under which spending decisions are independent of popularity and satisfy an equal ratio condition.

Our framework is flexible enough to allow for arbitrary initial advantages, early voting, candidates valuing money leftover at the end the campaign, multi-district competition, and the possibility of feedback between candidates’ popularity and their campaign budgets. We analyzed the robust prediction of our model, the equal spending ratio result, by looking at spending data from U.S. elections, and we recovered estimates of the perceived rate of decay of popularity leads.

To focus on the candidates’ budget allocation problems, we abstracted away from some important considerations in campaigning like the incentives of donors, and the candidates’

trade-off between campaigning and fundraising. These considerations are natural complements to our analysis.³⁹

We also abstracted from the fact that candidates may not know the return to spending or the decay rate of popularity leads at various stages of the campaign. These quantities may be specific to the characteristics of the candidates or to the political environment, including the “mood” of voters. Real-life candidates thus face an optimal experimentation problem whereby they try to learn about the campaign environment through early spending. There is no doubt that well-run campaigns spend resources to acquire valuable information about how voters are engaging with and responding to the candidates over time. These are interesting and important questions that ought to be addressed in subsequent work.

³⁹Mattozzi and Michelucci (2017) analyze a two-period dynamic model in which donors decide how much to contribute to each of two possible candidates without knowing ex-ante who is the more likely winner. Bouton et al. (2018) study the strategic choice of donors who try to affect the electoral outcome and highlights that donors’ behavior depends on the competitiveness of the election.

Appendix

A Proofs

A.1 Proof of Proposition 3

For the periods \hat{N}, \dots, N , we can write

$$\begin{aligned}
Z_{N\Delta} &= \left(1 - e^{-\lambda\Delta}\right) \sum_{n=0}^{N-1} e^{-\lambda\Delta(N-1-n)} p(x_{n\Delta}, y_{n\Delta}) + z_0 e^{-\lambda\Delta N} + \sum_{n=0}^{N-1} e^{-\lambda\Delta(N-1-n)} \varepsilon_{n\Delta}, \\
Z_{(N-1)\Delta} &= \left(1 - e^{-\lambda\Delta}\right) \sum_{n=0}^{N-2} e^{-\lambda\Delta(N-2-n)} p(x_{n\Delta}, y_{n\Delta}) + z_0 e^{-\lambda\Delta(N-1)} + \sum_{n=0}^{N-2} e^{-\lambda\Delta(N-2-n)} \varepsilon_{n\Delta}, \\
&\vdots \\
Z_{\hat{N}\Delta} &= \left(1 - e^{-\lambda\Delta}\right) \sum_{n=0}^{\hat{N}-1} e^{-\lambda\Delta(\hat{N}-1-n)} p(x_{n\Delta}, y_{n\Delta}) + z_0 e^{-\lambda\Delta\hat{N}} + \sum_{n=0}^{\hat{N}-1} e^{-\lambda\Delta(\hat{N}-1-n)} \varepsilon_{n\Delta}.
\end{aligned}$$

Substituting these in the objective function of the candidates, we can rewrite it as:

$$\Pr \left[\sum_{m=0}^{N-\hat{N}} \xi^m Z_{(N-m)\Delta} \geq 0 \right] = \Pr \left[\sum_{m=0}^{N-\hat{N}} \xi^m E_{N-m} \geq - \sum_{m=0}^{N-\hat{N}} \xi^m B_{N-m} \right],$$

where

$$B_k := \left(1 - e^{-\lambda\Delta}\right) \sum_{n=0}^{k-1} e^{-\lambda\Delta(N-1-n)} p(x_{n\Delta}, y_{n\Delta}) + z_0 e^{-\lambda\Delta k}$$

and

$$E_k := \sum_{n=0}^{k-1} e^{-\lambda\Delta(N-1-n)} \varepsilon_{n\Delta}.$$

Because all E_k are sums of normally distributed shocks whose variances do not depend on candidates' spending, we can equivalently assume that candidate 1 maximizes, and 2 minimizes $\sum_{m=0}^{N-\hat{N}} \xi^m B_{N-m}$. Hence, candidate 1 maximizes, and 2 minimizes:

$$\begin{aligned}
&\sum_{k=0}^{N-(\hat{N}+1)} \left(\sum_{m=0}^k \xi^m e^{-\lambda\Delta(k-m)} \right) p(x_{(N-1-k)\Delta}, y_{(N-1-k)\Delta}) + \\
&\quad + \left(\sum_{m=0}^{N-\hat{N}} \xi^m e^{-\lambda\Delta(k-m)} \right) \sum_{n=0}^{\hat{N}-1} e^{-\lambda\Delta(\hat{N}-1-n)} p(x_{n\Delta}, y_{n\Delta}).
\end{aligned}$$

It is thus clear from this that the equal spending ratio holds.

Now, for any two consecutive periods both prior to period \hat{N} , after we cancel out the constant terms, the consecutive period spending ratio is the same as the one derived for Section 2.3, hence it is constant. Consider two consecutive periods $(\hat{N} + k)$ and $(\hat{N} + k + 1)$, with $k \in \{0, \dots, N - \hat{N} - 2\}$. Reasoning as in Section 2.3, we can equate the first order conditions for these two periods and get

$$x_{(\hat{N}+k)\Delta}^\beta \frac{\partial p(x_{(\hat{N}+k)\Delta}, y_{(\hat{N}+k)\Delta})}{\partial x} = e^{\lambda\Delta} \left(\frac{(e^{-\lambda\Delta}/\xi) - (e^{-\lambda\Delta}/\xi)^{N-\hat{N}-k-1}}{1 - (e^{-\lambda\Delta}/\xi)^{N-\hat{N}-k-1}} \right) \times \\ \times x_{(\hat{N}+k+1)\Delta}^\beta \frac{\partial p(x_{(\hat{N}+k+1)\Delta}, y_{(\hat{N}+k+1)\Delta})}{\partial x}.$$

Because the term in parentheses above is lower than 1, if we compare this equation with equation (5), we can show that the consecutive period spending ratio is now lower. In particular, using the same steps used to prove Proposition 2 in the main text, we get that the consecutive period spending ratio is

$$\hat{r}_{\hat{N}+k} = e^{-\frac{\lambda\Delta}{\beta}} \left(\frac{(e^{-\lambda\Delta}/\xi) - (e^{-\lambda\Delta}/\xi)^{N-\hat{N}-k-1}}{1 - (e^{-\lambda\Delta}/\xi)^{N-\hat{N}-k-1}} \right)^{-\frac{1}{\beta}}.$$

and the term in parentheses is lower than 1. Observe that the term in parentheses is decreasing in k . Therefore, for $k \geq 0$, $\hat{r}_{\hat{N}+k}$ will be decreasing in k since $\beta < 0$. For the same reason, the term in parenthesis is also increasing in $e^{-\lambda\Delta}/\xi$ and thus the consecutive period spending ratio is decreasing in ξ .

A.2 Proof of Proposition 4

Assume that $p(x, y) = q(x/y)$ for some strictly increasing and strictly quasiconcave function q . Let (\hat{x}^*, \hat{y}^*) be an equilibrium.

Consider period $(N - 1)\Delta$ and pick an arbitrary history $h_{(N-1)\Delta}$. Let $(\hat{x}_{m\Delta})_{m=0}^{N-2}$ and $(\hat{y}_{m\Delta})_{m=0}^{N-2}$ be the amounts spent by the two candidates along this history. Candidate 1 maximizes $\mathbb{E}[\mathbf{1}_{\{Z_T \geq 0\}} + \kappa_1 X_T \mid h_{(N-1)\Delta}]$ and candidate 2 maximizes $\mathbb{E}[(1 - \mathbf{1}_{\{Z_T \geq 0\}}) + \kappa_2 Y_T \mid h_{(N-1)\Delta}]$. Let

$$L(h_{(N-1)\Delta}) \equiv \sum_{m=0}^{N-2} e^{-\lambda\Delta(N-1-m)} q\left(\frac{\hat{x}_{m\Delta}^*}{\hat{y}_{m\Delta}^*}\right) + q\left(\frac{x(h_{(N-1)\Delta})}{y(h_{(N-1)\Delta})}\right) + \\ + z_0 e^{-\lambda N\Delta} + \sum_{m=0}^{N-2} e^{-\lambda\Delta(N-1-m)} \varepsilon_{m\Delta}.$$

Given that $\varepsilon_{(N-1)\Delta} \sim \mathcal{N}(0, \sigma^2\Delta)$, we conclude that $Z_T \mid h_{(N-1)\Delta} \sim \mathcal{N}(L(h_{(N-1)\Delta}), \sigma^2\Delta)$. Hence, the first order conditions of the candidates are:

$$\begin{aligned}\phi_{(0,1)}\left(-\frac{\hat{L}(h_{(N-1)\Delta})}{\sigma\sqrt{\Delta}}\right) \times \frac{q'\left(\frac{\hat{x}^*(h_{(N-1)\Delta})}{\hat{y}^*(h_{(N-1)\Delta})}\right) \frac{1}{\hat{y}^*(h_{(N-1)\Delta})}}{\sigma\sqrt{\Delta}} &= \kappa_1; \\ \phi_{(0,1)}\left(-\frac{\hat{L}(h_{(N-1)\Delta})}{\sigma\sqrt{\Delta}}\right) \times \frac{q'\left(\frac{\hat{x}^*(h_{(N-1)\Delta})}{\hat{y}^*(h_{(N-1)\Delta})}\right) \frac{\hat{x}^*(h_{(N-1)\Delta})}{(\hat{y}^*(h_{(N-1)\Delta}))^2}}{\sigma\sqrt{\Delta}} &= \kappa_2.\end{aligned}$$

where $\phi_{(0,1)}$ is the pdf of the standardized normal and $\hat{L}(h_{(N-1)\Delta})$ is equal to $L(h_{(N-1)\Delta})$ after replacing the term $\frac{x(h_{(N-1)\Delta})}{y(h_{(N-1)\Delta})}$ with $\frac{\hat{x}^*(h_{(N-1)\Delta})}{\hat{y}^*(h_{(N-1)\Delta})}$. Taking the ratio of these expressions, we get $x_{(N-1)\Delta}^*(h_{(N-1)\Delta})/y_{(N-1)\Delta}^*(h_{(N-1)\Delta}) = \kappa_2/\kappa_1$, which is independent of the history $h_{(N-1)\Delta}$. Furthermore:

$$y_{(N-1)\Delta}^*(h_{(N-1)\Delta}) = \frac{q'\left(\frac{\kappa_2}{\kappa_1}\right)}{\kappa_1\sigma\sqrt{\Delta}}\phi_{(0,1)}\left(-\frac{\hat{L}(h_{(N-1)\Delta}) + q\left(\frac{\kappa_2}{\kappa_1}\right)}{\sigma\sqrt{\Delta}}\right),$$

and $x_{(N-1)\Delta}^*(h_{(N-1)\Delta}) = (\kappa_2/\kappa_1)y_{(N-1)\Delta}^*(h_{(N-1)\Delta})$. Hence, both spending decisions are decreasing in the absolute value of $-\left[\hat{L}(h_{(N-1)\Delta}) + q(\kappa_2/\kappa_1)\right]/\sigma\sqrt{\Delta}$, which depends on the history.

Assume for the sake of the inductive argument that for all histories $h_{m\Delta}$ with $m \in \{n+1, n+2, \dots, N-1\}$, we have that in an interior equilibrium (i) $\hat{x}^*(h_{m\Delta})/\hat{y}^*(h_{m\Delta}) = \kappa_2/\kappa_1$, and (ii) spending decisions are given by:

$$\begin{aligned}\hat{y}^*(h_{m\Delta}) &= \frac{e^{-\lambda\Delta(N-1-m)}q'\left(\frac{\kappa_2}{\kappa_1}\right)}{\kappa_1\sigma\sqrt{\Delta}\sum_{m'=m}^{N-1}e^{-2\lambda\Delta(N-1-m')}} \times \phi_{(0,1)}\left(-\frac{L^*(h_{m\Delta})}{\sigma\sqrt{\Delta}\sum_{m'=m}^{N-1}e^{-2\lambda\Delta(N-1-m')}}\right), \\ \hat{x}^*(h_{m\Delta}) &= \frac{\kappa_2}{\kappa_1}\hat{y}^*(h_{m\Delta})\end{aligned}$$

where

$$L^*(h_{m\Delta}) \equiv \sum_{m'=0}^{m-1} e^{-\lambda\Delta(N-1-m')} \left[q\left(\frac{\hat{x}_{m'\Delta}^*}{\hat{y}_{m'\Delta}^*}\right) + \varepsilon_{m'\Delta} \right] + z_0 e^{-\lambda N\Delta} + \sum_{m'=m}^{N-1} e^{-\lambda\Delta(N-1-m')} q\left(\frac{\kappa_2}{\kappa_1}\right),$$

and $(\hat{x}_{m'\Delta}^*)_{m'=0}^{m-1}$ and $(\hat{y}_{m'\Delta}^*)_{m'=0}^{m-1}$ are the spending decision of candidates along history $h_{m\Delta}$. Obviously, these spending decisions are decreasing in the absolute value of $L(h_{m\Delta})$.

Consider period $t = n\Delta$ and pick an arbitrary history $h_{n\Delta}$. Since $(\varepsilon_{s\Delta})_{s=0}^{N-1}$ are independent and indentially distributed according to a $\mathcal{N}(0, \sigma^2\Delta)$ and (by the inductive hypothesis) the ratios of spending decision in subsequent periods are history independent

and equal to κ_2/κ_1 , we have that $Z_T | h_{n\Delta} \sim \mathcal{N}(L(h_{n\Delta}), \sigma^2 \Delta \sum_{j=n}^{N-1} e^{-2\lambda\Delta(N-1-j)})$, where

$$L(h_{n\Delta}) \equiv \sum_{m=0}^{n-1} e^{-\lambda\Delta(N-1-m)} \left[q \left(\frac{\hat{x}_{m\Delta}^*}{\hat{y}_{m\Delta}^*} \right) + \varepsilon_{m\Delta} \right] + z_0 e^{-\lambda N\Delta} + \\ + e^{-\lambda\Delta(N-1-n)} q \left(\frac{x(h_{n\Delta})}{y(h_{n\Delta})} \right) + \sum_{m=n+1}^{N-1} e^{-\lambda\Delta(N-1-m)} q \left(\frac{\kappa_2}{\kappa_1} \right)$$

and $(\hat{x}_{m\Delta}^*)_{m=0}^{n-1}, (\hat{y}_{m\Delta}^*)_{m=0}^{n-1}$ are the amount spent by candidates in periods $m < n$ along history $h_{n\Delta}$.

The first order conditions for an interior optimum are:

$$\kappa_1 = \frac{e^{-\lambda\Delta(N-1-n)} q' \left(\frac{\hat{x}^*(h_{n\Delta})}{\hat{y}^*(h_{n\Delta})} \right) \frac{1}{\hat{y}^*(h_{n\Delta})}}{\sigma \sqrt{\Delta \sum_{m=n}^{N-1} e^{-2\lambda\Delta(N-1-m)}}} \times \phi_{(0,1)} \left(- \frac{\hat{L}(h_{n\Delta})}{\sigma \sqrt{\Delta \sum_{j=n}^{N-1} e^{-2\lambda\Delta(N-1-j)}}} \right); \\ \kappa_2 = \frac{e^{-\lambda\Delta(N-1-n)} q' \left(\frac{\hat{x}^*(h_{n\Delta})}{\hat{y}^*(h_{n\Delta})} \right) \frac{\hat{x}^*(h_{n\Delta})}{(\hat{y}^*(h_{n\Delta}))^2}}{\sigma \sqrt{\Delta \sum_{j=n}^{N-1} e^{-2\lambda\Delta(N-1-j)}}} \times \phi_{(0,1)} \left(- \frac{\hat{L}(h_{n\Delta})}{\sigma \sqrt{\Delta \sum_{j=n}^{N-1} e^{-2\lambda\Delta(N-1-j)}}} \right),$$

where $\hat{L}(h_{n\Delta})$ is equal to $L(h_{n\Delta})$ after replacing the term $\frac{x(h_{n\Delta})}{y(h_{n\Delta})}$ with $\frac{\hat{x}^*(h_{n\Delta})}{\hat{y}^*(h_{n\Delta})}$.

Taking the ratio of these expressions, we get $\hat{x}^*(h_{n\Delta})/\hat{y}^*(h_{n\Delta}) = \kappa_2/\kappa_1$, which is independent of the past. Thus candidates' equilibrium spending decisions are given by:

$$\hat{y}^*(h_{n\Delta}) = \frac{e^{-\lambda\Delta(N-1-n)} q' \left(\frac{\kappa_2}{\kappa_1} \right)}{\kappa_1 \sigma \sqrt{\Delta \sum_{m=n}^{N-1} e^{-2\lambda\Delta(N-1-m)}}} \times \phi_{(0,1)} \left(- \frac{\hat{L}(h_{n\Delta})}{\sigma \sqrt{\Delta \sum_{j=n}^{N-1} e^{-2\lambda\Delta(N-1-j)}}} \right); \\ \hat{x}^*(h_{n\Delta}) = \frac{\kappa_2}{\kappa_1} \hat{y}^*(h_{n\Delta}).$$

Given the equilibrium condition $\hat{x}^*(h_{n\Delta})/\hat{y}^*(h_{n\Delta}) = \kappa_2/\kappa_1$, we conclude that $\hat{L}(h_{n\Delta}) = \zeta((\varepsilon_{m\Delta})_{m=0}^{n-1})$, where $(\varepsilon_{m\Delta})_{m=0}^{n-1}$ are the shocks occurred along history $h_{n\Delta}$. Hence, candidates' spending decisions are decreasing in the absolute value of $\zeta((\varepsilon_{m\Delta})_{m=0}^{n-1})$. The statement of the proposition follows by induction.

A.3 Proof of Proposition 5

Note that there cannot be an equilibrium in which both candidates spend an amount equal to 0 in some district in the same period—in this case, footnote 22 implies that either candidate would have an incentive to deviate and spend a positive amount securing victory with probability 1. Hence, the game ends in a defeat for any candidate that spends 0 in

any district in any period. Thus, in equilibrium spending must be interior (i.e., satisfy the first order conditions) for any district and any period.

We now prove the proposition by induction. Consider the final period. Fix $(z_{T-\Delta}^s)_{s=1}^S$ arbitrarily. Suppose candidates 1 and 2 have budgets $X_{T-\Delta}$ and $Y_{T-\Delta}$, respectively in the last period. Fix an equilibrium strategy profile $(x_{T-\Delta}^{s*}, y_{T-\Delta}^{s*})_{s=1}^S$. We will show that, if they have budgets $\vartheta X_{T-\Delta}$ and $\vartheta Y_{T-\Delta}$, then $(\vartheta x_{T-\Delta}^{s*}, \vartheta y_{T-\Delta}^{s*})_{s=1}^S$ is an equilibrium. This implies that the equilibrium payoff in the last period is determined by $(z_{T-\Delta}^s)_{s=1}^S$ and $X_{T-\Delta}/Y_{T-\Delta}$ only. Suppose otherwise. Without loss, assume that there is $(\tilde{x}_{T-\Delta}^s)_{s=1}^S$ satisfying $\sum_{s=1}^S \tilde{x}_{T-\Delta}^s \leq \vartheta X_{T-\Delta}$, that gives a higher probability of winning to candidate 1 given $(z_{T-\Delta, m}^s)_{s=1}^S$ and $(\vartheta y_{T-\Delta}^{s*})_{s=1}^S$. Since the distribution of $(Z_T^s)_{s=1}^S$ is determined by $(z_{T-\Delta}^s)_{s=1}^S$ and $(x_{T-\Delta}^s/y_{T-\Delta}^s)_{s=1}^S$ only, this means that the distribution of $(Z_T^s)_{s=1}^S$ given $(z_{T-\Delta}^s)_{s=1}^S$ and $(\tilde{x}_{T-\Delta}^s/\vartheta y_{T-\Delta}^{s*})_{s=1}^S$ is more favorable to candidate 1 than that given $(x_{T-\Delta}^s/y_{T-\Delta}^s)_{s=1}^S$. Moreover, $(\vartheta x_{T-\Delta}^{s*}/\vartheta y_{T-\Delta}^{s*})_{s=1}^S = (x_{T-\Delta}^{s*}/y_{T-\Delta}^{s*})_{s=1}^S$ and candidate 1 could spend $(\frac{1}{\vartheta} \tilde{x}_{T-\Delta}^s)_{s=1}^S$ when the budgets are $(X_{T-\Delta}, Y_{T-\Delta})$. Because $(x_{T-\Delta}^{s*}, y_{T-\Delta}^{s*})_{s=1}^S$ is an equilibrium, the distribution of $(Z_T^s)_{s=1}^S$ given $(z_{T-\Delta}^s)_{s=1}^S$ is more favorable to candidate 1 under $(x_{T-\Delta}^{s*}/y_{T-\Delta}^{s*})_{s=1}^S$ than under $(\frac{1}{\vartheta} \tilde{x}_{T-\Delta}^s/y_{T-\Delta}^{s*})_{s=1}^S = (\tilde{x}_{T-\Delta}^s/\vartheta y_{T-\Delta}^{s*})_{s=1}^S$. This establishes a contradiction.

Now, we prove the inductive step. The inductive hypothesis is that the continuation payoff for either candidate in period $t \in \mathcal{T}$ can be written as a function of only the budget ratio $X_{t+\Delta}/Y_{t+\Delta}$ and vector $(z_{t+\Delta}^s)_{s=1}^S$ and candidates spend a positive amount in each district and in each following period. We have to show that $x_t^s/X_t = y_t^s/Y_t$ in each district s . For all $\tau \in \mathcal{T}$, let $x_\tau := \sum_s x_\tau^s$, $y_\tau := \sum_s y_\tau^s$ and $z_\tau := (z_\tau^s)_{s=1}^S$. Let $V_{t+\Delta}(\frac{X_{t+\Delta}}{Y_{t+\Delta}}, z_{t+\Delta})$ denote the continuation payoff of candidate 1 starting in period $t + \Delta$. Candidate 1's objective is

$$\max_{(x_t)_{s=1}^S} \int V_{t+\Delta} \left(\frac{X_t - x_t}{Y_t - y_t}, z_{t+\Delta} \right) \phi_t \left(z_{t+\Delta} \mid \left(\frac{x_t^s}{y_t^s} \right)_{s=1}^S, z_t \right) dz_{t+\Delta}$$

where $\phi_t(\cdot)$ is the conditional distribution of the vector $z_{t+\Delta}$. For each district s , the first order conditions for an interior optimum for candidate 1 is then equal to

$$\begin{aligned} & \frac{1}{Y_t - y_t} \int \frac{\partial V_{t+\Delta}((X_t - x_t)/(Y_t - y_t), z_{t+\Delta})}{\partial (x_t^s/y_t^s)} \phi_t \left(z_{t+\Delta} \mid (x_t^s/y_t^s)_{s=1}^S, z_t \right) dz_{t+\Delta} = \\ & = \frac{1}{y_t^s} \int V_{t+\Delta} \left(\frac{X_t - x_t}{Y_t - y_t}, z_{t+\Delta} \right) \frac{\partial \phi_t \left(z_{t+\Delta} \mid (x_t^s/y_t^s)_{s=1}^S, z_t \right)}{\partial (x_t^s/y_t^s)} dz_{t+\Delta}. \end{aligned}$$

Similarly, the objective function for candidate 2 is

$$\min_{(y_t^s)_{s=1}^S} \int V_{t+\Delta} \left(\frac{X_t - x_t}{Y_t - y_t}, z_{t+\Delta} \right) \phi_t \left(z_{t+\Delta} \mid \left(\frac{x_t^s}{y_t^s} \right)_{s=1}^S, z_t \right) dz_{t+\Delta}$$

and the corresponding first order condition for each s is

$$\begin{aligned} & \frac{X_t - x_t}{(Y_t - y_t)^2} \int \frac{\partial V_{t+\Delta}((X_t - x_t)/(Y_t - y_t), z_{t+\Delta})}{\partial(x_t^s/y_t^s)} \phi_t(z_{t+\Delta} \mid (x_t^s/y_t^s)_{s=1}^S, z_t) dz_{t+\Delta} = \\ & = \frac{x_t^s}{(y_t^s)^2} \int V_{t+\Delta}\left(\frac{X_t - x_t}{Y_t - y_t}, z_{t+\Delta}\right) \frac{\partial \phi_t(z_{t+\Delta} \mid (x_t^s/y_t^s)_{s=1}^S, z_t)}{\partial(x_t^s/y_t^s)} dz_{t+\Delta}. \end{aligned}$$

Dividing the candidate 1's first order condition by candidate 2's, we have

$$\frac{X_t - x_t}{Y_t - y_t} = \frac{x_t^s}{y_t^s},$$

which implies $x_t^s/y_t^s = X_t/Y_t$ for all s .

A.4 Proof of Proposition 6

Consider an arbitrary period $t = n\Delta$. By the assumptions on the payoff structure and on the popularity processes we can write:

$$\begin{aligned} \mathbb{E}[u_1((Z_T^s)_{s=1}^S) \mid h_t] &= \sum_{s=1}^S w^s \mathbb{E}[Z_T^s \mid h_t] = \\ &= \sum_{s=1}^S \left[w^s \sum_{m=0}^{N-n-1} \frac{1 - e^{-\lambda^s \Delta}}{\lambda^s} e^{-(N-n-1-m)\lambda^s \Delta} q \left(\frac{x_{(n+m)\Delta}^s}{y_{(n+m)\Delta}^s} \right) + e^{-(N-n)\lambda^s \Delta} z_t^s \right]. \end{aligned}$$

For every realization of the random shock, this expression is increasing in each $x_{(n+m)\Delta}^s$ and decreasing in each $y_{(n+m)\Delta}^s$. Hence, we can assume that candidate 1 maximizes

$$\sum_{s=1}^S \left[w^s \sum_{m=0}^{N-n-1} \frac{1 - e^{-\lambda^s \Delta}}{\lambda^s} e^{-(N-n-1-m)\lambda^s \Delta} q \left(\frac{x_{(n+m)\Delta}^s}{y_{(n+m)\Delta}^s} \right) \right],$$

while candidate 2 minimizes it; see also Appendix B.1 for a discussion of this point. The leftover budgets are $X_t = X_0 - \sum_{\tau < t} \sum_{s=1}^S x_\tau^s$ and $Y_t = Y_0 - \sum_{\tau < t} \sum_{s=1}^S y_\tau^s$.

The first equality in the statement of the proposition follows from Proposition 5. To prove the second equality, consider candidate 2's problem from the period-0 perspective (the proof for candidate 1 then follows from the equal spending ratio result). Computing the first order necessary conditions for an interior optimum associated with spending in

period $m\Delta$ in districts s and s' , and equating these conditions yields

$$w^s \frac{1 - e^{-\lambda^s \Delta}}{\lambda^s} e^{-(N-1-m)\lambda^s \Delta} \frac{\partial q(x_{m\Delta}^s/y_{m\Delta}^s)}{\partial (x_{m\Delta}^s/y_{m\Delta}^s)} \frac{x_{m\Delta}^s}{(y_{m\Delta}^s)^2} =$$

$$w^{s'} \frac{1 - e^{-\lambda^{s'} \Delta}}{\lambda^{s'}} e^{-(N-1-m)\lambda^{s'} \Delta} \frac{\partial q(x_{m\Delta}^{s'}/y_{m\Delta}^{s'})}{\partial (x_{m\Delta}^{s'}/y_{m\Delta}^{s'})} \frac{x_{m\Delta}^{s'}}{(y_{m\Delta}^{s'})^2}.$$

Exploiting fact that $x_{n\Delta}^s/y_{n\Delta}^s = X_{n\Delta}/Y_{n\Delta}$ for all $n = 0, \dots, N-1$, the previous expression simplifies to

$$\frac{y_{m\Delta}^s}{y_{m\Delta}^{s'}} = \frac{w^s}{w^{s'}} \Lambda(\lambda^s, \lambda^{s'}, m) \quad \text{for every } s, s' \text{ and every } m,$$

where $\Lambda(\lambda^s, \lambda^{s'}, m)$ is defined in (9).

A.5 Proofs for Section 3.4

We will in fact prove a more general result than Proposition 7 under which we also characterize the stochastic path of spending over time for this extension.

Applying Itô's lemma, we can write the process governing the evolution of this ratio for this model as:

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = \mu_{XY}(z_t)dt + \sigma_X dW_t^X - \sigma_Y dW_t^Y, \quad (17)$$

where

$$\mu_{XY}(z_t) = (a - b)z_t + \sigma_Y^2 - \rho\sigma_X\sigma_Y.$$

and $\rho \geq 0$ is the covariance between W_t^X and W_t^Y . Hence, the instantaneous volatility of this process is $\sigma_{XY} = \sqrt{\sigma_X^2 + \sigma_Y^2 - \rho\sigma_X\sigma_Y}$. Therefore, if at time $t \in \mathcal{T}$ the candidates have an amount of available resources equal to X_t and Y_t and spend x_t and y_t , then $Z_{t+\Delta}$ conditional on the history h_t at time t is a normal random variable:

$$Z_{t+\Delta} | h_t \sim \mathcal{N}\left(\log\left(\frac{x_t}{y_t}\right) \frac{1 - e^{-\lambda\Delta}}{\lambda} + z_t e^{-\lambda\Delta}, \frac{\sigma^2(1 - e^{-2\lambda\Delta})}{2\lambda}\right),$$

and Itô's lemma implies that

$$\log\left(\frac{X_{t+\Delta}}{Y_{t+\Delta}}\right) | h_t \sim \mathcal{N}\left(\log\left(\frac{X_t - x_t}{Y_t - y_t}\right) + \mu_{XY}(z_t)\Delta, \sigma_{XY}^2\Delta\right).$$

Last, let $g_1(0) = 1$ and $g_2(0) = 0$, and define recursively for every $m \in \{1, \dots, N-1\}$,

$$\begin{pmatrix} g_1(m) \\ g_2(m) \end{pmatrix} = \begin{pmatrix} e^{-\lambda\Delta} & a - b \\ \frac{1 - e^{-\lambda\Delta}}{\lambda} & 1 \end{pmatrix} \begin{pmatrix} g_1(m-1) \\ g_2(m-1) \end{pmatrix}. \quad (18)$$

Then we have the following result, which implies Proposition 7 in the main text.

Proposition A.1. *Let $t = (N - m)\Delta \in \mathcal{T}$ be a time at which $X_t, Y_t > 0$. Then, in the essentially unique equilibrium, spending ratios are equal to*

$$\frac{x_t}{X_t} = \frac{y_t}{Y_t} = \frac{g_1(m-1)}{g_1(m-1) + g_2(m-1) \frac{\lambda}{1-e^{-\lambda\Delta}}}. \quad (19)$$

Moreover, in equilibrium, $(\log(x_{t+n\Delta}/y_{t+n\Delta}), z_{t+n\Delta}) \mid h_t$ follows a bivariate normal distribution with mean

$$\begin{pmatrix} 1 & (a-b)\Delta \\ \frac{1-e^{-\lambda\Delta}}{\lambda} & e^{-\lambda\Delta} \end{pmatrix}^n \begin{pmatrix} \log\left(\frac{X_t}{Y_t}\right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_t + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{pmatrix} - \begin{pmatrix} \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ \frac{\sigma_Y^2 - \rho\sigma_X\sigma_Y}{a-b} \end{pmatrix}$$

and variance

$$\begin{pmatrix} 1 & (a-b)\Delta \\ \frac{1-e^{-\lambda\Delta}}{\lambda} & e^{-\lambda\Delta} \end{pmatrix}^n \begin{pmatrix} \sigma_{XY}^2 \Delta & 0 \\ 0 & \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda} \end{pmatrix} \begin{pmatrix} 1 & \frac{1-e^{-\lambda\Delta}}{\lambda} \\ (a-b)\Delta & e^{-\lambda\Delta} \end{pmatrix}^n.$$

Proof. Consider time $t = n\Delta \in \mathcal{T}$ and suppose that at time t both candidates have still a positive budget, $X_t, Y_t > 0$. We will prove the proposition by induction on the times at which candidates take actions, $t = (N - m)\Delta \in \mathcal{T}$, $m = 1, 2, \dots, N$.

To simplify notation, let $g_1(0) = 1, g_2(0) = 0, g_3(0) = 0$ and $g_4(0) = 0$. Furthermore, using (18), recursively write for every $m \in \{1, 2, \dots, N\}$,

$$\begin{aligned} g_3(m) &= g_2(m-1)\Delta + g_3(m-1) \\ g_4(m) &= (g_1(m-1))^2 \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda} + (g_2(m-1))^2 \sigma_{XY}^2 \Delta + g_4(m-1) \end{aligned}$$

Diagonalizing the matrix in (18) and solving for $(g_1(m), g_2(m))'$ with initial conditions $g_0(1) = 1$ and $g_2(0) = 0$, we can conclude that, for each $N \in \mathbb{N}$ and $\lambda, \Delta > 0$, there exists $-\eta < 0$ such that, if $a - b \geq -\eta$, both $g_1(m)$ and $g_2(m)$ are non-negative for each m . In the proof, we will thus assume that $g_1(m) \geq 0$ and $g_2(m) \geq 0$ for every $m = 1, \dots, N$.

The inductive hypothesis is the following: for every $\tau = (N - m)\Delta \in \mathcal{T}$, $m \in \{1, \dots, N\}$, if $X_\tau, Y_\tau > 0$, then

- (i) the continuation payoff of each candidate is a function of current popularity z_τ , current budget ratio X_τ/Y_τ and calendar time τ ;

(ii) the distribution of Z_T given z_τ and X_τ/Y_τ is $\mathcal{N}\left(\hat{\mu}_{(N-m)\Delta}(z_\tau), \hat{\sigma}_{(N-m)\Delta}^2\right)$, where

$$\begin{aligned}\hat{\mu}_{(N-m)\Delta}(z_{(N-m)\Delta}) &= g_1(m)z_{(N-m)\Delta} + g_2(m) \log\left(\frac{X_{(N-m)\Delta}}{Y_{(N-m)\Delta}}\right) + g_3(m)(\sigma_Y^2 - \rho\sigma_X\sigma_Y), \\ \hat{\sigma}_{(N-m)\Delta}^2 &= g_4(m).\end{aligned}$$

Base Step Consider $m = 1$, the subgame reached in the final period $t = (N - 1)\Delta$ and suppose both candidates still have a positive amount of resources, $X_{(N-1)\Delta}, Y_{(N-1)\Delta} > 0$. Both candidates will spend their remaining resources: $x_{(N-1)\Delta} = X_{(N-1)\Delta}$ and $y_{(N-1)\Delta} = Y_{(N-1)\Delta}$. Hence, $x_{(N-1)\Delta}/y_{(N-1)\Delta} = X_{(N-1)\Delta}/Y_{(N-1)\Delta}$ and

$$Z_T | h_{(N-1)\Delta} \sim \mathcal{N}\left(\log\left(\frac{X_{(N-1)\Delta}}{Y_{(N-1)\Delta}}\right) \frac{1 - e^{-\lambda\Delta}}{\lambda} + z_{(N-1)\Delta} e^{-\lambda\Delta}, \frac{\sigma^2(1 - e^{-2\lambda\Delta})}{2\lambda}\right).$$

Because Z_T fully determines the candidates' payoffs, the continuation payoff of the candidates is a function of current popularity $z_{(N-1)\Delta}$, the ratio $X_{(N-1)\Delta}/Y_{(N-1)\Delta}$, and calendar time. Furthermore, given the recursive definition of g_1, g_2, g_3 and g_4 , we can conclude that the second part of the inductive hypothesis also holds at $t = (N - 1)\Delta$.

Inductive Step Suppose the inductive hypothesis holds true at any time $(N - m)\Delta \in \mathcal{T}$ with $m \in \{1, 2, \dots, m^* - 1\}$, $m^* \leq N$. We want to show that at time $(N - m^*)\Delta \in \mathcal{T}$, if $X_t, Y_t > 0$, then (i) an equilibrium exists, (ii) in all equilibria, $x_t/y_t = X_t/Y_t$ and the continuation payoffs of both candidates are functions of relative popularity z_t , the ratio X_t/Y_t , and calendar time t , and (iii) Z_T given period t information is distributed according to $\mathcal{N}\left(\hat{\mu}_{(N-m^*)\Delta}(z_t), \hat{\sigma}_{(N-m^*)\Delta}^2\right)$.

Consider period $t = N - m^*$ and let $x_t, y_t > 0$ be the candidates' spending in this period. Exploiting the inductive hypothesis, the distribution of $Z_{t+\Delta} | h_t$ and the one of $\log\left(\frac{X_{t+\Delta}}{Y_{t+\Delta}}\right) | h_t$, we can compound the normal distributions and conclude that $Z_T | h_t \sim \mathcal{N}(\tilde{\mu}, \tilde{\sigma}^2)$, where

$$\begin{aligned}\tilde{\mu} &= \hat{\mu}_t(x_t, y_t) := G_1 \log\left(\frac{x_t}{y_t}\right) + G_2 \log\left(\frac{X_{(N-m^*)\Delta} - x_t}{Y_{(N-m^*)\Delta} - y_t}\right) + G_3, \\ \tilde{\sigma}^2 &= G_4,\end{aligned}$$

with G_1, G_2, G_3 and G_4 defined as follows:

$$G_1 = g_1(m^* - 1) \frac{1 - e^{-\lambda\Delta}}{\lambda}, \quad (20)$$

$$G_2 = g_2(m^* - 1), \quad (21)$$

$$G_3 = g_1(m^* - 1)z_t e^{-\lambda\Delta} + g_2(m^* - 1)\mu_{XY}(z_t)\Delta + g_3(m^* - 1)(\sigma_Y^2 - \rho\sigma_X\sigma_Y), \quad (22)$$

$$G_4 = (g_1(m^* - 1))^2 \frac{\sigma^2(1 - e^{-2\lambda\Delta})}{2\lambda} + (g_2(m^* - 1))^2 \sigma_{XY}^2 \Delta + g_4(m^* - 1). \quad (23)$$

Note that $\tilde{\sigma}^2$ is independent of x_t and y_t .

Candidate 1 wins the election if $Z_T > 0$. Thus, in equilibrium he chooses x_t to maximize his winning probability

$$\int_{-\frac{\hat{\mu}_t(x_t, y_t)}{\tilde{\sigma}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds.$$

The first order necessary condition for x_t is given by

$$\frac{1}{\sqrt{2\pi}} e^{\frac{\hat{\mu}_t(x_t, y_t)}{2\tilde{\sigma}_t}} \frac{\hat{\mu}'_t(x_t, y_t)}{\tilde{\sigma}} = \frac{1}{\sqrt{2\pi}\tilde{\sigma}} e^{\frac{\hat{\mu}_t(x_t, y_t)}{2\tilde{\sigma}}} \left[\frac{G_1(X_t - x_t) - G_2 x_t}{x_t(X_t - x_t)} \right].$$

Furthermore, when the first order necessary condition holds, the second order condition is given by

$$\frac{1}{\sqrt{2\pi}} e^{\frac{\hat{\mu}_t(x_t, y_t)}{2\tilde{\sigma}_t}} \frac{\hat{\mu}''_t(x_t, y_t)}{\tilde{\sigma}} = \frac{-1}{\sqrt{2\pi}} e^{\frac{\hat{\mu}_t(x_t, y_t)}{2\tilde{\sigma}}} \left[\frac{G_1(X_t - x_t)^2 + G_2 x_t^2}{x_t^2(X_t - x_t)^2} \right] < 0.$$

Hence, the problem is strictly quasi-concave for candidate 1 for each y_t . A symmetric argument shows that the corresponding problem for candidate 2 is strictly quasi-concave for each x_t . Hence an equilibrium exists and the optimal investment of the two candidates is pinned down by the first order necessary conditions, which yields

$$\frac{x_t}{X_t} = \frac{y_t}{Y_t} = \frac{G_1}{G_1 + G_2}. \quad (24)$$

Thus, in equilibrium, $x_t/y_t = X_t/Y_t$ and $(X_t - x_t)/(Y_t - y_t) = X_t/Y_t$. Because the continuation payoffs of candidates is fully determined by Z_T , these expected payoffs from the perspective of time t depend only on calendar time, the level of current popularity and the ratio of budget at time t . Furthermore, recalling the definition of $\mu_{XY}(z_t)$, we conclude that the second part of the inductive hypothesis is also true.

Next, we know that

$$Z_T \mid h_{(N-m^*)\Delta} \sim \mathcal{N}(\hat{\mu}_{(N-m^*)\Delta}, \hat{\sigma}_{(N-m^*)\Delta}^2),$$

where

$$\hat{\mu}_{(N-m^*)\Delta}(z_{(N-m^*)\Delta}) = g_1(m^*)z_{(N-m^*)\Delta} + g_2(m^*) \log \left(\frac{X_{(N-m^*)\Delta}}{Y_{(N-m^*)\Delta}} \right) + g_3(m^*)(\sigma_Y^2 - \rho\sigma_X\sigma_Y),$$

$$\hat{\sigma}_{(N-m^*)\Delta}^2 = g_4(m^*).$$

The expression for x_t/X_t and y_t/Y_t in the proposition thus follows from (18), (20), (21) and (24). To derive the distribution of $(x_t/y_t, z_t)$, we first use the proof of Proposition 7 to derive the distribution of $x_{t+j\Delta}/y_{t+j\Delta}$ and $z_{t+j\Delta}$ given x_t/y_t and z_t . Let

$$\Sigma = \begin{pmatrix} \sigma_{XY}^2 \Delta & 0 \\ 0 & \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda} \end{pmatrix}.$$

Because $X_t/Y_t = x_t/y_t$ for each t , we can write

$$\left(\begin{array}{c} \log \left(\frac{x_{t+n\Delta}}{y_{t+n\Delta}} \right) \\ z_{t+n\Delta} \end{array} \right) \left| \left(\begin{array}{c} \frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}} \\ z_{t+(n-1)\Delta} \end{array} \right) \right. \sim \mathcal{N} \left(\left(\begin{array}{c} \log \left(\frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}} \right) + \mu_{XY}(z_{t+(n-1)\Delta})\Delta \\ \log \left(\frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}} \right) \frac{1-e^{-\lambda\Delta}}{\lambda} + z_{t+(n-1)\Delta}e^{-\lambda\Delta} \end{array} \right), \Sigma \right).$$

Define

$$A = \begin{pmatrix} 1 & (a-b)\Delta \\ \frac{1-e^{-\lambda\Delta}}{\lambda} & e^{-\lambda\Delta} \end{pmatrix}.$$

and notice that the previous distribution implies

$$\left(\begin{array}{c} \log \left(\frac{x_{t+n\Delta}}{y_{t+n\Delta}} \right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_{t+n\Delta} + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{array} \right) \left| \left(\begin{array}{c} \log \left(\frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}} \right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_{t+(n-1)\Delta} + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{array} \right) \right.$$

follows a multivariate normal distribution

$$\mathcal{N} \left(A \left(\begin{array}{c} \log \left(\frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}} \right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_{t+(n-1)\Delta} + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{array} \right), \Sigma \right).$$

Therefore, we conclude that

$$\left(\begin{array}{c} \log \left(\frac{x_{t+n\Delta}}{y_{t+n\Delta}} \right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_{t+n\Delta} + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{array} \right) \left| \left(\begin{array}{c} \log \left(\frac{x_t}{y_t} \right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_t + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{array} \right) \right.$$

follows the multivariate normal distribution

$$\mathcal{N} \left(A^n \begin{pmatrix} \log \left(\frac{X_t}{Y_t} \right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_t + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{pmatrix}, A^n \Sigma (A^T)^n \right).$$

■

Proof of Proposition 8. Fix λ and Δ . and let $n = N - m$. We must show that for all $n \in \{0, \dots, N - 1\}$,

$$\check{r}_n(a - b) = \frac{x_{n\Delta}}{X_{n\Delta}} / \frac{x_{(n+1)\Delta}}{X_{(n+1)\Delta}}$$

is decreasing in $\alpha := a - b$ around $\alpha = 0$. Note that \check{r}_n is the same as \check{r}_{N-m} .

Proposition A.1 and (18) imply

$$\check{r}_m(\alpha) = \frac{g_1(m-1) \left(g_1(m) + g_2(m) \frac{\lambda}{1-e^{-\lambda\Delta}} \right)}{\left(g_1(m-1) + g_2(m-1) \frac{\lambda}{1-e^{-\lambda\Delta}} \right) g_1(m)} = \frac{g_1(m-1) g_2(m+1)}{g_1(m) g_2(m)}.$$

Furthermore, (18) also implies

$$g_1(m) = \frac{(\lambda + \alpha) e^{-\lambda\Delta} - \alpha}{\lambda} g_1(m-1) + \alpha g_2(m), \quad (25)$$

$$g_2(m+1) = \frac{(1 - e^{-\lambda\Delta}) ((\lambda + \alpha) e^{-\lambda\Delta} - \alpha)}{\lambda^2} g_1(m-1) + \frac{\alpha - \alpha e^{-\lambda\Delta} + \lambda}{\lambda} g_2(m). \quad (26)$$

Substituting in the expression for $\check{r}_m(\alpha)$ and simplifying, we get

$$\check{r}_m(\alpha) = \frac{1}{\frac{(\lambda + \alpha) e^{-\lambda\Delta} - \alpha}{\lambda} + \alpha g_m} \left(\frac{(1 - e^{-\lambda\Delta}) ((\lambda + \alpha) e^{-\lambda\Delta} - \alpha)}{\lambda^2} \frac{1}{g_m} + \frac{\alpha - \alpha e^{-\lambda\Delta} + \lambda}{\lambda} \right), \quad (27)$$

where $g_m := g_2(m) / g_1(m-1)$. We can thus identify two values of g_m for which (27) holds. However, if α is sufficiently low, namely if $\alpha < \lambda / (1 + e^{\lambda\Delta})$, one of these two values is negative and thus not feasible. Thus, if α is sufficiently small, (27) enables us to express g_m as a function of $\check{r}_m(\alpha)$. Moreover, from (25) and (26), we further have

$$g_{m+1} = \frac{\frac{1-e^{-\lambda\Delta}}{\lambda} \frac{(\lambda+\alpha)e^{-\lambda\Delta}-\alpha}{\lambda} + \frac{\alpha+\lambda-\alpha e^{-\lambda\Delta}}{\lambda} g_m}{\frac{(\lambda+\alpha)e^{-\lambda\Delta}-\alpha}{\lambda} + \alpha g_m}. \quad (28)$$

Computing (27) one step forward and substituting for g_{m+1} as obtained from (28) and, subsequently, for g_m as obtained from (27), we get \check{r}_{m+1} as a function of α and \check{r}_m , written $\check{r}_{m+1}(\alpha, \check{r}_m)$.

Given the expression for \check{r}_{m+1} , we can show by induction that $\check{r}_m > e^{\lambda\Delta} > 1$ for each m around $\alpha = 0$. When $m = 1$, we have $x_{(N-1)\Delta}/X_{(N-1)\Delta} = 1$ and $x_{(N-2)\Delta}/X_{(N-2)\Delta} = g_1(1) / \left(g_1(1) + g_2(1) \frac{\lambda}{1-e^{-\lambda\Delta}} \right)$. Substituting for $g_1(1)$ and $g_2(1)$, we get $\check{r}_1 - e^{\lambda\Delta} = 1$. Thus, $\check{r}_1 > e^{\lambda\Delta} > 1$. Suppose $\check{r}_m > e^{\lambda\Delta} > 1$. Then, subtracting $e^{\lambda\Delta}$ from the right hand side of the expression of \check{r}_{m+1} and setting $\alpha = 0$, we get $\check{r}_{m+1} - e^{\lambda\Delta} = 1 - e^{\lambda\Delta}/\check{r}_m > 0$. We conclude that, if $\check{r}_m > e^{\lambda\Delta}$, then $\check{r}_{m+1} > e^{\lambda\Delta}$ in a neighborhood of $\alpha = 0$. Therefore, $\check{r}_m > e^{\lambda\Delta}$ for each m in a neighborhood of $\alpha = 0$.

Furthermore, $\check{r}_{m+1}(\alpha, \check{r}_m)$ is decreasing in α and increasing in \check{r}_m at $\alpha = 0$:

$$\begin{aligned} \left. \frac{\partial \check{r}_{m+1}(\alpha, \check{r}_m)}{\partial \alpha} \right|_{\alpha=0} &= - \frac{(\check{r}_m - 1) e^{\lambda\Delta} (e^{2\lambda\Delta} - 1)}{\check{r}_m (\check{r}_m - e^{\lambda\Delta})} < 0; \\ \left. \frac{\partial \check{r}_{m+1}(\alpha, \check{r}_m)}{\partial \check{r}_m} \right|_{\alpha=0} &= \frac{e^{\lambda\Delta}}{(\check{r}_m)^2} > 0. \end{aligned}$$

Hence, a simple induction argument implies that $\check{r}_m(\alpha)$ is decreasing in α for each m in a neighborhood of $\alpha = 0$.

Finally, \check{r}_m is increasing in λ as well:

$$\left. \frac{\partial \check{r}_{m+1}(\alpha, \check{r}_m, \lambda)}{\partial \lambda} \right|_{\alpha=0} = \frac{e^{\lambda\Delta} (\check{r}_m - 1) \Delta}{\check{r}_m} > 0 \text{ for each } \lambda > 0.$$

Thus, a symmetric inductive argument shows that \check{r}_m is increasing in λ for every m in a neighborhood of $\alpha = 0$. ■

Finally, we prove that the equal spending ratio result holds for the specification of the model described in the final paragraph of Section 3.4.

Proposition A.2. *In the model with endogenous budgets that evolve depending on the closeness of the race, if for all $t \in \mathcal{T}$, $X_t, Y_t > 0$, then in equilibrium,*

$$x_t/X_t = y_t/Y_t.$$

Proof. For any $t \in \mathcal{T}$ the distribution of $Z_{t+\Delta} | h_t$ is given by (2), while Ito's lemma implies:

$$\log \left(\frac{X_{t+\Delta}}{Y_{t+\Delta}} \right) | h_t \sim \mathcal{N} \left(\log \left(\frac{X_t - x_t}{Y_t - y_t} \right) + m(z_t), \sigma_{XY}^2 \Delta \right), \quad (29)$$

where $m_{XY}(z_t) = (a-b)/(1+z_t^2) + \sigma_Y^2 - \rho\sigma_X\sigma_Y$ and $\sigma_{XY}^2 = \sigma_X^2 + \sigma_Y^2 - \rho\sigma_X\sigma_Y$. Furthermore, the two distributions are independent (conditional on h_t). Let ϕ_1 and ϕ_2 be the pdfs of these two distributions. The proof is by induction.

Base Step. Consider period $t = (N - 1)\Delta$. Because money leftover has no value and we are considering an interior equilibrium, $x_{(N-1)\Delta} = X_{(N-1)\Delta}$ and $y_{(N-1)\Delta} = Y_{(N-1)\Delta}$. Thus, the equal spending holds at time $t = (N - 1)\Delta$. Also, observe that the continuation payoff of candidates is fully determined by the distribution of Z_T and, in equilibrium,

$$Z_T | h_{(N-1)\Delta} \sim \mathcal{N} \left(\log \left(\frac{X_{(N-1)\Delta}}{Y_{(N-1)\Delta}} \right) \frac{1 - e^{-\lambda\Delta}}{\lambda} + z_{(N-1)\Delta} e^{-\lambda\Delta}, \frac{\sigma^2(1 - e^{-2\lambda\Delta})}{2\lambda} \right).$$

Hence, in equilibrium, the expected continuation payoff of candidates at time $(N - 1)\Delta$ depends on the popularity at time $(N - 1)\Delta$, $z_{(N-1)\Delta}$, and on the logarithm of the available budgets, $\log(X_{(N-1)\Delta}/Y_{(N-1)\Delta})$. Denote an expected continuation payoff for candidate 1 by $V_{(N-1)\Delta}(z_{(N-1)\Delta}, X_{(N-1)\Delta}/Y_{(N-1)\Delta})$. Obviously, the expected continuation payoff for candidate 2 is $1 - V_{(N-1)\Delta}(z_{(N-1)\Delta}, X_{(N-1)\Delta}/Y_{(N-1)\Delta})$.

Inductive Step. Pick $m \in \{0, \dots, N - 2\}$ and suppose that for all periods $\tau \in \{(N - m + 1)\Delta, (N - m + 2)\Delta, \dots, (N - 1)\Delta\}$ in an interior equilibrium the equal spending ratio result holds and the expected continuation payoff of candidates depends on z_τ , and on X_τ and Y_τ only through the log of their ratio, $\log(X_\tau/Y_\tau)$. Denote this continuation for candidate 1 with $V_\tau(z_\tau, X_\tau/Y_\tau)$. Then, at time $t = (N - m)\Delta$, the expected payoff of candidate 1 is:

$$V_t(z_t, x_t, y_t) = \int \phi_1(z_{t+\Delta} | z_t, x_t, y_t) \phi_2 \left(z_{t+\Delta}, \log \left(\frac{X_{t+\Delta}}{Y_{t+\Delta}} \right) | z_t, x_t, y_t \right) \times \\ \times V \left(z_{t+\Delta}, \log \left(\frac{X_{t+\Delta}}{Y_{t+\Delta}} \right) \right) d(z_{t+\Delta}, X_{t+\Delta}, Y_{t+\Delta}). \quad (30)$$

Candidate 1 chooses x_t to maximize $V_t(z_t, x_t, y_t)$ and candidate 2 chooses y_t to minimize it. Hence the two first order conditions are given by:

$$\begin{aligned} \frac{1}{x_t} \int \frac{\partial \phi_1}{\partial \mu_1} \phi_2 V_{t+\Delta} d(z_{t+\Delta}, X_{t+\Delta}, Y_{t+\Delta}) = \\ + \frac{1}{X_t - x_t} \int \phi_1 \left(\frac{\partial \phi_2}{\partial \mu_2} V_{t+\Delta} + \phi_2 \frac{\partial V_{t+\Delta}}{\partial \log \left(\frac{X_t - x_t}{Y_t - y_t} \right)} \right) d(z_{t+\Delta}, X_{t+\Delta}, Y_{t+\Delta}); \\ \frac{1}{y_t} \int \frac{\partial \phi_1}{\partial \mu_1} \phi_2 V_{t+\Delta} d(z_{t+\Delta}, X_{t+\Delta}, Y_{t+\Delta}) = \\ + \frac{1}{Y_t - y_t} \int \phi_1 \left(\frac{\partial \phi_2}{\partial \mu_2} V_{t+\Delta} + \phi_2 \frac{\partial V_{t+\Delta}}{\partial \log \left(\frac{X_t - x_t}{Y_t - y_t} \right)} \right) d(z_{t+\Delta}, X_{t+\Delta}, Y_{t+\Delta}). \end{aligned}$$

Consider candidate 1 (the reasoning for candidate 2 is identical). Spending 0 at time t is not compatible with equilibrium behavior: if candidate 1 spends 0 at time t , a deviation to

spending $X_\tau/(N - \tau)$ in all periods $\tau \geq t$ would strictly increase the winning probability.⁴⁰ Hence both candidates must be spending a positive amount in period t . Similarly, $x_t = X_t$ cannot be compatible with equilibrium behavior either: if candidate 2 is spending a positive amount in period $(N - m + 1)\Delta$, this strategy would lead to the defeat of candidate 1 in period $(N - m + 1)\Delta$, while by spending $x_\tau = X_\tau/(N - \tau)$ for all $\tau \geq t$ candidate 1 could win with positive probability. (If candidate 2 is spending 0 in period $(N - m + 1)\Delta$, $x_t = X_t$ would lead candidate 1 to win with probability 1/2, while $x_\tau = X_\tau/(N - \tau)$ for all $\tau \geq t$ would guarantee victory with probability 1.) Hence the equilibrium must be interior and the first order condition must hold. Thus, taking the ratio of the two first order conditions, we get the equal spending ratio result. Hence, in an interior equilibrium $x_t/y_t = (X_t - x_t)/(Y_t - y_t) = X_t/Y_t$. Furthermore, the expected continuation payoff in an interior equilibrium depends on the popularity z_t and on the initial budgets X_t, Y_t only through $\log(X_t/Y_t)$. ■

A.6 Proof of Proposition 10

We prove the statement of the proposition only for candidate 1 because the analysis for candidate 2 is identical. Let $\lambda > 0$ and $\gamma_n^\Delta := x_{n\Delta}^\Delta/X_0$ and $\gamma_n^{\bar{\Delta}} := x_{n\bar{\Delta}}^{\bar{\Delta}}/X_0$. If $M = 1$, the result holds trivially. Thus, suppose $M > 1$ and fix any $t = n\Delta \in \mathcal{T}$ with $n \leq N - 1$. Then,

$$\begin{aligned} \sum_{m=0}^{M-1} x_{t+m\bar{\Delta}}^{\bar{\Delta}} &= X_0 \sum_{m=0}^{M-1} \gamma_{nM+m}^{\Delta/M} = X_0 \frac{e^{\lambda\Delta/M} - 1}{e^{\lambda NM\Delta/M} - 1} \sum_{m=0}^{M-1} e^{\lambda\Delta(nM+m)/M} = \\ &= X_0 \frac{e^{\lambda\Delta/M} - 1}{e^{\lambda N\Delta} - 1} e^{\lambda\Delta n} \frac{e^{\lambda\Delta} - 1}{e^{\lambda\Delta/M} - 1} = X_0 \frac{e^{\lambda\Delta} - 1}{e^{\lambda N\Delta} - 1} e^{\lambda\Delta n} = X_0 \gamma_n^\Delta = x_t^\Delta. \end{aligned}$$

The proof for the case in which $\lambda = 0$ is similar and omitted.

B General Setting

The equilibrium of the baseline model in the main text exhibits two main features. First, candidates' spending decisions are independent of the popularity process: $x_{n\Delta}$ and $y_{n\Delta}$ are not a function of $(z_{m\Delta})_{m \leq n}$. Second, the equal spending ratio result holds: $x_{n\Delta}/X_{n\Delta} = y_{n\Delta}/Y_{n\Delta}$ for every $n = 1, \dots, N - 1$. These properties immediately implies that the ratio of nominal spending $x_{n\Delta}/y_{n\Delta}$ is constant across histories and equal to X_0/Y_0 .

In this appendix, we provide sufficient conditions for each of these two properties to hold in a more general contest setting. We outline two approaches. In the first approach, we make a monotone separability assumption that allows us to cast the stochastic problem

⁴⁰The probability would jump from 0 to a positive amount if candidate 2 was spending a positive amount and from 1/2 to 1 if candidate 2 was spending 0.

faced by candidates as a deterministic problem, which does not depend on past realizations of the popularity shock (Section B.1). This approach is different from methods grounded in the “value function approach” (cf. Seel and Strack, 2016, Hörner, 2004). On the other hand, when the separability assumption does not hold, we can still characterize the equilibrium spending path by taking the ratio of first order conditions without actually having to solve these conditions (Section B.2). This second approach requires additional assumptions on the function $p(x, y)$, which captures the impact of spending decisions on the drift of the popularity process.

Consider a game where two players, 1 and 2, allocate their fixed resources across the N periods in \mathcal{T} . Player 1’s vector of spending is $(x_{n\Delta})_{n=0}^{N-1}$, while player 2’s is $(y_{n\Delta})_{n=0}^{N-1}$. Let h_t denote the history of the game up to time $t \in \mathcal{T}$. The state variable of the contest in period n is $S_{n\Delta} \equiv (z_{n\Delta}, X_{n\Delta}, Y_{n\Delta})$, where $X_{n\Delta}$ is the resource stock available to player 1, $Y_{n\Delta}$ is the resource stock available to player 2, and $z_{n\Delta}$ measures the advantage of player 1. The state at the final period T is $S_T \equiv (z_T, X_T, Y_T)$, where X_T is player 1’s resources left over and Y_T is player 2’s. Player 1’s payoff at the end of the game is denoted with $u_1(z_T)$ and we assume that it is strictly increasing in z_T . Similarly, Player 2’s payoff at the end of the game is denoted with $u_2(z_T)$ and we assume that it is strictly decreasing in z_T .

$Z_{n\Delta}$ evolves a stochastic process, whose law of motion is

$$Z_{(n+1)\Delta} = F_{n\Delta}(z_{n\Delta}, x_{n\Delta}, y_{n\Delta}, \varepsilon_{n\Delta})$$

with $\varepsilon_{n\Delta}$ being a random i.i.d. shock with CDF Φ . Without loss of generality let $\mathbb{E}[\varepsilon_{n\Delta} | h_{n\Delta}] = 0$. Further, we assume throughout that the collection $(F_{n\Delta})_{n=0}^{N-1}$ is made of twice continuously differentiable functions, and that the partial derivatives of each $F_{n\Delta}$ with respect to $x_{n\Delta}$ (respectively, $y_{n\Delta}$) is positive (respectively, negative).

This model nests the baseline model in the main text, if we assume that the process $F_{n\Delta}(z_{n\Delta}, x_{n\Delta}, y_{n\Delta}, \varepsilon_{n\Delta})$ is given by equation (2).

B.1 When is spending independent of the state of the contest?

In the baseline model, the amount spent by players in equilibrium is independent of the past realizations of the popularity process and thus the players’ spending ratios are also popularity-independent. To understand when this condition holds in the general setting we have introduced, consider the following assumption.

Assumption B.1. *For every $n = 0, \dots, N - 1$, $F_{n\Delta}$ is linearly separable among terms that depend on $Z_{n\Delta}$, $(x_{n\Delta}, y_{n\Delta})$ and $\varepsilon_{n\Delta}$ — specifically, for parameter ω_n and function p_n ,*

$$Z_{(n+1)\Delta} = \omega_n Z_{n\Delta} + p_n(x_{n\Delta}, y_{n\Delta}) + \varepsilon_{n\Delta}$$

Assumption B.1 implies that $(Z_{n\Delta})_{n \geq 0}$ is an AR(1) process. Furthermore, when this assumption holds Z_T can be expanded as

$$Z_T = \Omega_0 z_0 + \sum_{n=0}^{N-1} \Omega_{n+1} p_n(x_{n\Delta}, y_{n\Delta}) + \sum_{n=0}^{N-1} \Omega_{n+1} \varepsilon_{n\Delta}, \quad (31)$$

where $\Omega_{N-1} = 1$ and for every $n < N - 1$, $\Omega_n := \prod_{m=n}^{N-1} \omega_m$. Therefore

$$\mathbb{E}[u_i(Z_T) | h_{n\Delta}] = \mathbb{E} \left[u_i \left(\Omega_{n\Delta} z_{n\Delta} + \sum_{m=0}^{N-1} \Omega_{m+1} p_m(x_{m\Delta}, y_{m\Delta}) + \sum_{m=0}^{N-1} \Omega_{m+1} \varepsilon_{m\Delta} \right) \middle| h_{n\Delta} \right].$$

The next assumption is a monotone separability condition. It says that the players' expected utilities at any history $h_{n\Delta}$ can be written as a function that is separable into three arguments—a first that depends on the p_n terms, a second that depends on the shock terms $\varepsilon_{n\Delta}$ and a third that depends on the initial value of Z —and is monotone in the first.

Assumption B.2. *For each player i and for every time $n\Delta \in \mathcal{T}$ there exist functions $U_{i,n}$ and a function \tilde{u}_i such that*

$$\mathbb{E}[u_i(Z_T) | h_{n\Delta}] = U_{i,n} \left(\tilde{u}_i \left((x_{m\Delta})_{m=0}^{N-1}, (y_{m\Delta})_{m=0}^{N-1} \right), (\varepsilon_{m\Delta})_{m=0}^{n-1}, z_0 \right),$$

and $U_{i,n}$ is monotone in the first argument in the same direction for every n and for every vector of shock realizations $(\varepsilon_{m\Delta})_{m=0}^{n-1}$.

Under this assumption, let $\iota_i = 1$ denote the case in which $U_{i,n}$ is increasing in the first argument and $\iota_i = -1$ the case in which it is decreasing. The assumption implies that a profile of actions $(x_{m\Delta}^*, y_{m\Delta}^*)_{m=0}^{N-1}$ is a state independent equilibrium if and only if $(x_{m\Delta}^*)_{m=0}^{N-1}$ and $(y_{m\Delta}^*)_{m=0}^{N-1}$ respectively solve

$$\max \iota_1 \tilde{u}_1 \left((x_{m\Delta})_{m=0}^{N-1}, (y_{m\Delta})_{m=0}^{N-1} \right) \text{ subject to } \sum_{m=0}^{N-1} x_{m\Delta} \leq X_0 \quad (32)$$

$$\max \iota_2 \tilde{u}_2 \left((x_{m\Delta})_{m=0}^{N-1}, (y_{m\Delta})_{m=0}^{N-1} \right) \text{ subject to } \sum_{m=0}^{N-1} y_{m\Delta} \leq Y_0 \quad (33)$$

We say that \tilde{u}_1 satisfies the Inada-0 conditions in $x_{n\Delta}$ if $\lim_{x_{n\Delta} \rightarrow 0} \tilde{u}_1 = \infty$ and \tilde{u}_2 satisfies the Inada-0 conditions in $y_{n\Delta}$ if $\lim_{y_{n\Delta} \rightarrow 0} \tilde{u}_2 = \infty$. The next result follows immediately from the assumptions.

Theorem B.1. *Suppose Assumptions B.1 and B.2 hold. Then, there is an equilibrium that is independent of $(Z_{n\Delta})_{n \geq 0}$. If, in addition, for all periods n , \tilde{u}_1 is strictly quasiconcave and satisfies the Inada-0 conditions in each $x_{n\Delta}$ and \tilde{u}_2 is strictly quasiconcave and satisfies the Inada-0 conditions in each $y_{n\Delta}$, then this equilibrium is unique.*

Note that in our baseline model in section 2, $\varepsilon_{n\Delta}$ is a normal shock, $u_1(S_T) = +\mathbf{1}_{\{Z_T \geq 0\}}$ and $u_2(S_T) = -\mathbf{1}_{\{Z_T \geq 0\}}$. In this case, Assumption B.1 is satisfied and implies that

$$\begin{aligned} \mathbb{E}[u_1(S_T) | h_{n\Delta}] &= \Pr(Z_T \geq 0 | h_{n\Delta}) = \\ &= 1 - \Phi \left(-\frac{\sum_{m=0}^{N-1} \Omega_{m+1} p_m(x_{m\Delta}, y_{m\Delta})}{\sum_{m=n}^{N-1} \Omega_{m+1}} - \frac{\sum_{m=0}^{n-1} \Omega_{m+1} \varepsilon_{m\Delta}}{\sum_{m=n}^{N-1} \Omega_{m+1}} \right) \end{aligned}$$

This function is monotone in $\sum_{m=0}^{N-1} \Omega_{m+1} p_m(x_{m\Delta}, y_{m\Delta})$ for every $(\varepsilon_{m\Delta})_{m=0}^{n-1}$, and the analogue is true for Player 2. Thus Assumption B.2 is also satisfied. Since $p(\cdot, y)$ and $p(x, \cdot)$ are respectively strictly quasiconcave and strictly quasiconvex, this implies that the optimal allocation is independent of the history of the contest. This result holds in other cases as well, as the next example shows.

Example 1. Suppose $u_1(S_T) = Z_T$ and $u_2(S_T) = -Z_T$. This example captures the case where the players care linearly about their margin of victory in the contest. Then, under Assumptions B.1,

$$\mathbb{E}[u_1(S_T) | h_{n\Delta}] = \sum_{m=0}^{N-1} \Omega_{m+1} p_m(x_{m\Delta}, y_{m\Delta}) + \sum_{m=0}^{n-1} \Omega_{m+1} \varepsilon_{m\Delta},$$

This expression is monotone in $\sum_{m=0}^{N-1} \Omega_{m+1} p_m(x_{m\Delta}, y_{m\Delta})$ for every $\sum_{m=0}^{n-1} \Omega_{m+1} \varepsilon_{m\Delta}$. The analogue is true for player 2. And, in fact, the same would be true if one of the players cared about the margin, but the other only cared about winning, since the monotone separability condition would follow exactly as in the baseline model. In either case, Assumption B.2 holds and we can conclude that players' equilibrium allocations are independent of the history of Z , provided p_m are all strictly quasiconcave in $x_{m\Delta}$ and strictly quasiconvex in $y_{m\Delta}$.

If Assumption B.2 fails, the equilibrium choices at time t may depend on the past shocks. We give an example of this in Section 3.2 of the main text. When this is the case, the characterization of the equilibrium is analytically challenging. Nonetheless, the first-order approach may still yield a constant ratio of nominal spending. This could be sufficient to characterize the final outcome of the contest. We investigate this next.

B.2 When does the first-order approach characterize the spending path?

In Section 2.3 we characterize the equilibrium spending path by taking the ratio of the players' first order conditions to show that the equal spending ratio result holds. To what extent can we generalize this approach? When does it generate the equal spending ratio result, and what can we say about the equilibrium spending path when it does not? To answer these questions we assume throughout this section that an interior equilibrium exists

and that the solution to the first-order necessary conditions is unique.⁴¹ We also introduce the following assumption.

Assumption B.3. *There is an invertible function $\psi : (0, \infty) \rightarrow \mathbb{R}$ s.t. for all periods n*

$$\forall x_{n\Delta}, y_{n\Delta} > 0, \quad \frac{\partial F_{n\Delta}/\partial x_{n\Delta}}{\partial F_{n\Delta}/\partial y_{n\Delta}} = \psi\left(\frac{x_{n\Delta}}{y_{n\Delta}}\right).$$

Assumption B.3 guarantees that marginal changes in the players' investments affect the evolution of Z_t not through their absolute values, but only through their ratio.⁴² Under the payoffs of the baseline model, this assumption amounts to saying that the preferences of the players are homothetic. Note also that, within the baseline model, Assumption B.3 holds when spending decisions affect the evolution of $Z_{(n+1)\Delta}$ only through their ratio, that is when $p(x, y) = q(x/y)$ for some strictly increasing and strictly concave function q . Thus Assumption B.3 generalizes the *ratio scale invariance* assumption introduced in Section 3.2. Finally, Assumption B.3 is satisfied if the law of motion of Z depends on players' spending through the Tullock function $x/(x + y)$. We then have the following result.⁴³

Theorem B.2. *Suppose that $u_1(Z_T) = v(Z_T)$ and $u_2(Z_T) = -v(Z_T)$, where $v(Z_T)$ is almost everywhere differentiable, nonconstant, and weakly increasing functions of Z_T . Then, if Assumption 3 holds, in any equilibrium the equal spending ratio result holds: $x_t/X_t = y_t/Y_t$ for all $t \in \mathcal{T}$ s.t. $X_t, Y_t > 0$. As a result, the ratio of nominal spending is constant over time.*

The assumption on payoffs stated in Theorem B.2 implies that the game is zero sum. Thus, player 2 minimizes player 1's objective function (and vice versa). When this is the case and Assumption B.3 holds, the equal spending ratio result can be established by taking the ratio of first order conditions for the two players in any given period without actually solving these conditions. Each player equalizes the marginal benefit of spending today with the marginal benefit of spending in the final period. These marginals take the same form for the two players. Furthermore, their ratio can be inverted by Assumption B.3. Therefore, the ratio of player 1's spending in the current period to him spending final period is equal to the same ratio for player 2. The equal spending ratio result follows from this.

Proof of Theorem B.2. We start introducing some notation. Let $(z(h_t), X(h_t), Y(h_t))$ be the state vector at history h_t . Let (\hat{x}, \hat{y}) denote a profile of strategies. $\hat{x}(h)$ and $\hat{y}(h)$ are the actions prescribed by these strategies at non-terminal history h . Finally, we write $h' \preceq h''$ to denote that history h' precedes history h'' .

⁴¹In the baseline model, this was guaranteed by equation (4) and the properties of function p .

⁴²This assumption holds in the specification of Section 2.3, by defining $\psi(x/y) = -(\alpha_1/\alpha_2)(x/y)^\beta$.

⁴³Although the game is zero-sum, the proof of Theorem B.2(i) hinges on candidates moving simultaneously and independently within each period. The reason why Proposition 1(ii) does not generalize to this setting is that, under the general structure of this Appendix, a player's optimal spending at time t may depend on the other player's past allocation decisions.

By recursive substitution in the law of motion, we get

$$\begin{aligned} Z_T &= F_{(N-1)\Delta}(Z_{(N-1)\Delta}, x_{(N-1)\Delta}, y_{(N-1)\Delta}, \varepsilon_{(N-1)\Delta}) \\ &= F_{(N-1)\Delta}(F_{(N-2)\Delta}, x_{(N-2)\Delta}, y_{(N-2)\Delta}, \varepsilon_{(N-2)\Delta}), x_{(N-1)\Delta}, y_{(N-1)\Delta}, \varepsilon_{(N-1)\Delta}) \\ &\vdots \end{aligned}$$

Suppose Assumption B.3 holds. Player 1 maximizes $\mathbb{E}[v(Z_T)]$, while player 2 minimizes it. Furthermore both players spend the entirety of their remaining budgets at any history $h_{(N-1)\Delta}$. Hence, for every $h_{(N-1)\Delta}$ with $h_{(N-2)\Delta} \preceq h_{(N-1)\Delta}$, the equilibrium $x^*(h_{(N-1)\Delta})$ and $y^*(h_{(N-1)\Delta})$ must satisfy (here we assume that history $h_{(N-1)\Delta}$ includes the spending choices of candidates at history $h_{(N-2)\Delta}$, $x(h_{(N-2)\Delta})$ and $y(h_{(N-2)\Delta})$)

$$\begin{aligned} x^*(h_{(N-1)\Delta}) &= X(h_{(N-2)\Delta}) - x(h_{(N-2)\Delta}); \\ y^*(h_{(N-1)\Delta}) &= Y(h_{(N-2)\Delta}) - y(h_{(N-2)\Delta}). \end{aligned}$$

Consider any history $h_{(N-2)\Delta}$. The previous equalities imply

$$\frac{\partial x^*(h_{(N-1)\Delta})}{\partial x_{(N-2)\Delta}} = \frac{\partial y^*(h_{(N-1)\Delta})}{\partial y_{(N-2)\Delta}} = -1.$$

Since players move simultaneously and independently, we also have:

$$\frac{\partial y^*(h_{(N-1)\Delta})}{\partial x_{(N-2)\Delta}} = \frac{\partial x^*(h_{(N-1)\Delta})}{\partial y_{(N-2)\Delta}} = 0.$$

The first order conditions for player 1 is

$$\begin{aligned} 0 = \mathbb{E} \left[v'(Z_T) \left(\frac{\partial F_{(N-1)\Delta}}{\partial Z_{(N-1)\Delta}} \frac{\partial F_{(N-2)\Delta}}{\partial x_{(N-2)\Delta}} + \frac{\partial F_{(N-1)\Delta}}{\partial x_{(N-1)\Delta}} \frac{\partial x^*(h_{(N-1)\Delta})}{\partial x_{(N-2)\Delta}} + \right. \right. \\ \left. \left. + \frac{\partial F_{(N-1)\Delta}}{\partial y_{(N-1)\Delta}} \frac{\partial y^*(h_{(N-1)\Delta})}{\partial x_{(N-1)\Delta}} \right) \mid h_{(N-2)\Delta} \right] \end{aligned}$$

and using the previous results it simplifies to:

$$\mathbb{E} \left[v'(Z_T) \frac{\partial F_{(N-1)\Delta}}{\partial Z_{(N-1)\Delta}} \frac{\partial F_{(N-2)\Delta}}{\partial x_{(N-2)\Delta}} \mid h_{(N-2)\Delta} \right] = \mathbb{E} \left[v'(Z_T) \frac{\partial F_{(N-1)\Delta}}{\partial x_{(N-1)\Delta}} \mid h_{(N-2)\Delta} \right].$$

Similarly, the first order condition for Player 2 can be written as:

$$\mathbb{E} \left[v'(Z_T) \frac{\partial F_{(N-1)\Delta}}{\partial Z_{(N-1)\Delta}} \frac{\partial F_{(N-2)\Delta}}{\partial y_{(N-2)\Delta}} \mid h_{(N-2)\Delta} \right] = \mathbb{E} \left[v'(Z_T) \frac{\partial F_{(N-1)\Delta}}{\partial y_{(N-1)\Delta}} \mid h_{(N-2)\Delta} \right].$$

Taking the ratio of these first order conditions, exploiting Assumption B.3, and using again the fact that the entire budget is exhausted we get that equilibrium spending decision at history $h_{(N-2)\Delta}$, $(x^*(h_{(N-2)\Delta}), y^*(h_{(N-2)\Delta}))$, must satisfy $x^*(h_{(N-2)\Delta})/y^*(h_{(N-2)\Delta}) = [X(h_{(N-2)\Delta}) - x^*(h_{(N-2)\Delta})]/[Y(h_{(N-2)\Delta}) - y^*(h_{(N-2)\Delta})]$. Rearranging this last expression, we get $x^*(h_{(N-2)\Delta})/X(h_{(N-2)\Delta}) = y^*(h_{(N-2)\Delta})/Y(h_{(N-2)\Delta})$. Obviously, the ratio of nominal spending at history $h_{(N-2)\Delta}$, $x^*(h_{(N-2)\Delta})/y^*(h_{(N-2)\Delta})$, does not depend on the popularity process.

Suppose for the sake of induction that for any history $h_{(n+1)\Delta}$ in the unique equilibrium (x^*, y^*) : (a) for every history h with $h_{(n+1)\Delta} \lesssim h$, $x^*(h)/X(h) = y^*(h)/Y(h)$, and (b) for every history $h_{m\Delta}$ up to time $m\Delta$ with $m \geq (n+1)$, the following two first order conditions hold

$$\begin{aligned} \mathbb{E} \left[v'(Z_T) \frac{\partial F_{(N-1)\Delta}}{\partial Z_{(N-1)\Delta}} \cdots \frac{\partial F_{(m+1)\Delta}}{\partial Z_{(m+1)\Delta}} \frac{\partial F_{m\Delta}}{\partial x_{m\Delta}} \mid h_{m\Delta} \right] &= \mathbb{E} \left[v'(Z_T) \frac{\partial F_{(N-1)\Delta}}{\partial x_{(N-1)\Delta}} \mid h_{m\Delta} \right]; \\ \mathbb{E} \left[v'(Z_T) \frac{\partial F_{(N-1)\Delta}}{\partial Z_{(N-1)\Delta}} \cdots \frac{\partial F_{(m+1)\Delta}}{\partial Z_{(m+1)\Delta}} \frac{\partial F_{m\Delta}}{\partial y_{m\Delta}} \mid h_{m\Delta} \right] &= \mathbb{E} \left[v'(Z_T) \frac{\partial F_{(N-1)\Delta}}{\partial y_{(N-1)\Delta}} \mid h_{m\Delta} \right]. \end{aligned}$$

These conditions imply that for every $h_{m\Delta}$ with $m = n+1, \dots, N-1$, $x^*(h_{m\Delta})/y^*(h_{m\Delta})$ does not depend on $Z_{m\Delta}$.

Pick an arbitrary history $h_{n\Delta}$. The first order condition for candidate 1 is:

$$\begin{aligned} 0 &= \mathbb{E} \left[v'(Z_T) \frac{\partial F_{(N-1)\Delta}}{\partial Z_{(N-1)\Delta}} \cdots \frac{\partial F_{(n+1)\Delta}}{\partial Z_{(n+1)\Delta}} \frac{\partial F_{n\Delta}}{\partial x_{n\Delta}} \mid h_{n\Delta} \right] \\ &+ \mathbb{E} \left[v'(Z_T) \sum_{m=n+1}^{N-2} \left(\frac{\partial F_{(N-1)\Delta}}{\partial Z_{(N-1)\Delta}} \cdots \frac{\partial F_{(m+1)\Delta}}{\partial Z_{(m+1)\Delta}} \frac{\partial F_{m\Delta}}{\partial x_{m\Delta}} \right) \frac{\partial x^*(h_{m\Delta})}{\partial x_{n\Delta}} \mid h_{n\Delta} \right] \\ &+ \mathbb{E} \left[v'(Z_T) \sum_{m=n+1}^{N-2} \left(\frac{\partial F_{(N-1)\Delta}}{\partial Z_{(N-1)\Delta}} \cdots \frac{\partial F_{(m+1)\Delta}}{\partial Z_{(m+1)\Delta}} \frac{\partial F_{m\Delta}}{\partial y_{m\Delta}} \right) \frac{\partial y^*(h_{m\Delta})}{\partial x_{n\Delta}} \mid h_{n\Delta} \right] \\ &+ \mathbb{E} \left[v'(Z_T) \frac{\partial F_{(N-1)\Delta}}{\partial x_{(N-1)\Delta}} \frac{\partial x^*(h_{(N-1)\Delta})}{\partial x_{n\Delta}} \mid h_{n\Delta} \right] + \mathbb{E} \left[v'(Z_T) \frac{\partial F_{(N-1)\Delta}}{\partial y_{(N-1)\Delta}} \frac{\partial y^*(h_{(N-1)\Delta})}{\partial x_{n\Delta}} \mid h_{n\Delta} \right]. \end{aligned}$$

Because money left over is not valuable, both players exhaust their endowment by the end of the campaign. Thus:

$$\frac{\partial x^*(h_{(N-1)\Delta})}{\partial x_{n\Delta}} = -1 - \sum_{m=n+1}^{N-2} \frac{\partial x^*(h_{m\Delta})}{\partial x_{n\Delta}} \quad \text{and} \quad \frac{\partial y^*(h_{(N-1)\Delta})}{\partial x_{n\Delta}} = - \sum_{m=n+1}^{N-2} \frac{\partial y^*(h_{m\Delta})}{\partial x_{n\Delta}}$$

where $h_{m\Delta} \prec h_{(N-1)\Delta}$ for every m . (Once more, the second equality uses the fact that players move simultaneously and independently so $x_{n\Delta}$ has no effect on player's 2 choice at time $n\Delta$.) Therefore, from the law of iterated expectations and condition (b) in the

inductive hypothesis, we can rewrite first order conditions as follows:

$$\mathbb{E} \left[v'(Z_T) \frac{\partial F_{(N-1)\Delta}}{\partial Z_{(N-1)\Delta}} \cdots \frac{\partial F_{(n+1)\Delta}}{\partial Z_{(n+1)\Delta}} \frac{\partial F_{n\Delta}}{\partial x_{n\Delta}} \mid h_{n\Delta} \right] = \mathbb{E} \left[v'(Z_T) \frac{\partial F_{(N-1)\Delta}}{\partial x_{(N-1)\Delta}} \mid h_{n\Delta} \right].$$

An analogous argument for candidate 2 yields

$$\mathbb{E} \left[v'(Z_T) \frac{\partial F_{(N-1)\Delta}}{\partial Z_{(N-1)\Delta}} \cdots \frac{\partial F_{(n+1)\Delta}}{\partial Z_{(n+1)\Delta}} \frac{\partial F_{n\Delta}}{\partial y_{n\Delta}} \mid h_{n\Delta} \right] = \mathbb{E} \left[v'(Z_T) \frac{\partial F_{(N-1)\Delta}}{\partial y_{(N-1)\Delta}} \mid h_{n\Delta} \right].$$

Hence, condition (b) of the inductive hypothesis holds true also at time $n\Delta$. If we take the ratio of these first order conditions and use Assumption B.3 and condition (a) in the inductive hypothesis, we obtain:

$$\psi \left(\frac{x^*(h_{n\Delta})}{y_{n\Delta}} \right) = \psi \left(\frac{X(h_{n\Delta}) - x^*(h_{n\Delta}) - \sum_{m=n+1}^{N-2} x^*(h_{m\Delta})}{Y(h_{n\Delta}) - y^*(h_{n\Delta}) - \sum_{m=n+1}^{N-2} y^*(h_{m\Delta})} \right).$$

This yields $x^*(h_{n\Delta})/X(h_{n\Delta}) = y^*(h_{n\Delta})/Y(h_{n\Delta})$ and concludes the inductive step (note that the ratio of nominal spending is independent of the popularity process). The result follows by induction. \blacksquare

C Quantitative Analysis of Campaign Silence Regulations

Here, we provide a quantitative analysis of campaign silence regulations that prohibit campaign spending in the final seven days of a campaign, as well as the effects of static spending caps. Our focus is on the expected win rate of the high quality candidate in the setting studied in Section 4 of the main text. For our analysis, we simulate elections over a 12-week campaign period with decay rates ranging between 0% and 90%.

We let candidate 1 be the high quality candidate, and use the functional forms governing popularity defined in Section 4.2, setting $c_{1,1} = c_{2,2} = 0.5$, $c_{1,2} = c_{2,1} = 0.4$, $c = 0.5$, and let $\beta \in \{0.5, 0.4, 0.3\}$. The parameters $c_{i,j}$ ($i, j \in \{0, 1\}$) determine the relative effectiveness of the ads on the core supporters of the two candidates, and β governs the ‘instantaneous’ effect of campaign spending on influencing voters. We assume a laissez faire benchmark spending level of \$500,000 for each candidate, and a static cap of \$250,000. As Figure C.1 shows, the point at which campaign silence becomes welfare improving is lower, if campaign spending is more effective. Static caps are welfare decreasing for all parameter values.

The benefit of campaign silence over the laissez faire benchmark becomes larger in settings with higher decay rates. In Figure C.1, the black vertical line represents the decay rate value at which imposing a campaign silence policy becomes welfare improving. Under campaign silence, higher decay rates imply a bigger shift of spending from later weeks to earlier weeks, reducing the influence of impressionable voters on the outcome.

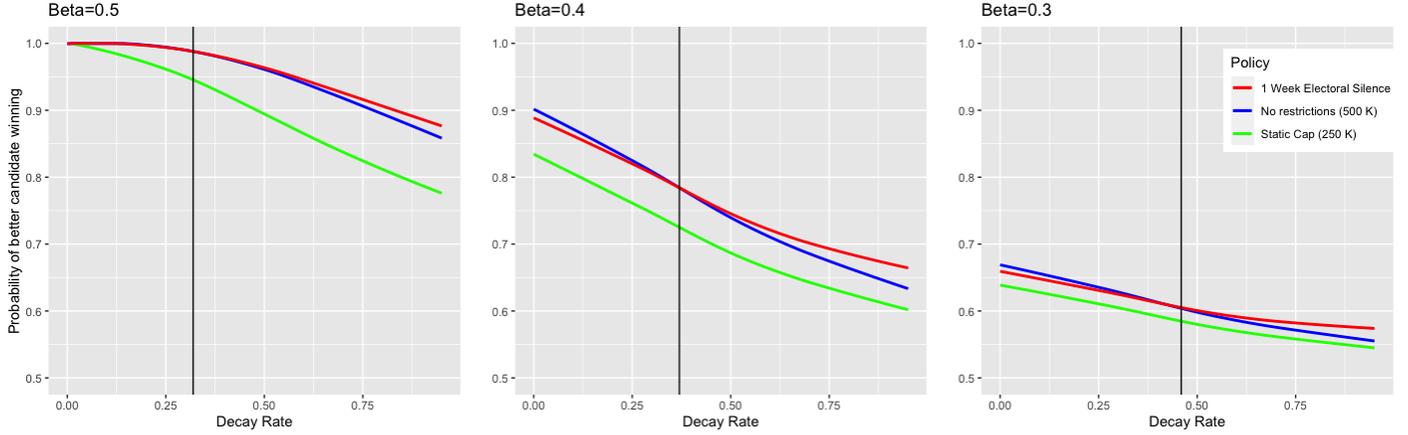


Figure C.1: The probability of the higher quality candidate winning under alternative campaign spending regulations. In each plot, the black vertical line marks the decay rate value at which a one week campaign silence policy becomes welfare improving, compared to the unregulated case.

D Diagnostics

D.1 Spending Ratios

To investigate the extent to which the equal spending ratio result holds in the data, we set $\Delta = 1$ week and plot the difference $x_t/X_t - y_t/Y_t$ over the final twelve weeks of each election in Figure D.1 and we tabulate the percent of elections, by election type, in which each candidate’s spending was within 10 percentage points of the other’s in the upper panel of Table D.1.⁴⁴ The absolute difference in spending ratios is less than 0.1 for 85% of our dataset. Even in the final six weeks where all candidates spend a positive amount, the candidates’ spending ratios are within 10 percentage points of one another in about 75% of election-weeks. The lower panel of Table D.1 reports the percent of elections by week in which each candidate’s spending was within 5 percentage points of the other’s. The absolute difference in spending is less than 0.05 for 65% of the data overall, and in the final six weeks it is less than 0.05 in half the elections. Overall, Table D.1 shows that the equal spending ratio prediction seems to be violated to a smaller extent in statewide races than in House races, and violated to a greater extent as election day approaches. So, while there are violations of the equal spending ratio result, the spending ratios of the two candidates are remarkably close for the majority of these election-weeks.

⁴⁴Note that since these values are defined as the share of *remaining* budget rather than *total* budget, they can take any value between 0 and 1 in every week in the data prior to the final week. (For example, a candidate can be spending 99% of their remaining budget in every week until the final week.) In the final week, each candidate spends 100% of money left over, so if we added the final (partial) twelfth week of the election, to the final column, these numbers would all be 100%, by construction.

Table D.1: $x_t/X_t - y_t/Y_t$

Week	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
% $\in (-0.1, 0.1)$	0.963	0.953	0.938	0.902	0.879	0.847	0.829	0.754	0.676	0.622	0.797
Senate	0.943	0.934	0.975	0.926	0.934	0.885	0.844	0.787	0.746	0.648	0.803
Governor	0.932	0.910	0.887	0.820	0.812	0.812	0.767	0.774	0.639	0.624	0.782
House	0.983	0.977	0.945	0.925	0.884	0.847	0.847	0.734	0.665	0.613	0.801
Early Voting	0.970	0.955	0.942	0.912	0.884	0.844	0.816	0.753	0.673	0.612	0.798
No Early Voting	0.951	0.951	0.931	0.882	0.868	0.853	0.853	0.755	0.681	0.642	0.794
Open Seat	0.942	0.933	0.920	0.897	0.857	0.862	0.866	0.795	0.705	0.656	0.804
Incumbent Competing	0.976	0.966	0.950	0.905	0.891	0.838	0.806	0.729	0.658	0.602	0.793
Close Election	0.976	0.965	0.935	0.941	0.947	0.924	0.906	0.882	0.776	0.706	0.788
Not Close Election	0.958	0.949	0.940	0.886	0.852	0.817	0.798	0.703	0.636	0.589	0.800
Close Budgets	0.974	0.974	0.959	0.925	0.914	0.895	0.883	0.812	0.763	0.695	0.838
Not Close Budgets	0.955	0.937	0.922	0.884	0.851	0.809	0.785	0.707	0.606	0.564	0.764
% $\in (-0.05, 0.05)$	0.865	0.815	0.757	0.727	0.661	0.599	0.554	0.468	0.418	0.369	0.562
Senate	0.811	0.762	0.664	0.762	0.713	0.648	0.639	0.566	0.492	0.393	0.598
Governor	0.782	0.744	0.759	0.639	0.586	0.519	0.489	0.481	0.406	0.346	0.556
House	0.916	0.861	0.789	0.749	0.671	0.613	0.549	0.428	0.396	0.370	0.552
Early Voting	0.864	0.814	0.763	0.746	0.660	0.602	0.542	0.463	0.406	0.370	0.562
No Early Voting	0.868	0.819	0.745	0.691	0.662	0.593	0.578	0.475	0.441	0.368	0.564
Open Seat	0.799	0.799	0.723	0.763	0.688	0.625	0.603	0.549	0.460	0.362	0.580
Incumbent Competing	0.905	0.825	0.777	0.706	0.645	0.584	0.525	0.419	0.393	0.374	0.552
Close Election	0.853	0.841	0.841	0.812	0.741	0.729	0.706	0.553	0.535	0.424	0.576
Not Close Election	0.870	0.805	0.724	0.694	0.629	0.548	0.494	0.434	0.371	0.348	0.557
Close Budgets	0.880	0.842	0.789	0.793	0.741	0.677	0.628	0.526	0.515	0.474	0.590
Not Close Budgets	0.854	0.794	0.731	0.675	0.597	0.537	0.496	0.421	0.340	0.287	0.540
Average x_t/X_t	0.021 (0.032)	0.028 (0.036)	0.039 (0.044)	0.054 (0.051)	0.075 (0.054)	0.109 (0.067)	0.134 (0.073)	0.184 (0.085)	0.251 (0.095)	0.377 (0.108)	0.728 (0.076)
Average y_t/Y_t	0.021 (0.035)	0.029 (0.041)	0.038 (0.046)	0.049 (0.053)	0.074 (0.063)	0.105 (0.073)	0.133 (0.080)	0.184 (0.094)	0.249 (0.097)	0.380 (0.111)	0.733 (0.073)

Note: The table reports the share of elections in which spending ratios are within 0.1 and 0.05 for every week, across different election types. We define close elections to be races where the final difference in vote shares between two candidates is less than 5 percentage points. We define races in which the budgets are close to be races where the ratio of budgets of the two candidates are in the interval (0.75, 1.25).

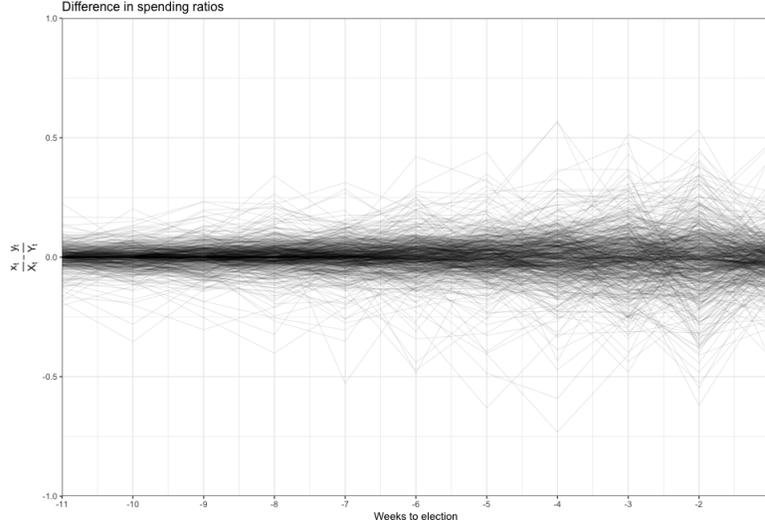


Figure D.1: The difference in spending ratios between the Democratic candidate (x_t/X_t) and the Republican candidate (y_t/Y_t) for each week in our dataset. Each line is an election.

In addition, Table D.1 looks separately at (i) elections with early voting versus those without, (ii) those that are open seat versus those in which an incumbent is running, (iii) those in which the final vote difference between the top two candidates is less than 5 percentage points versus those with larger margins, and (iv) those in which one candidate’s budget is more than 25 percent greater than the other’s, versus those where it is not. We do not find major differences in the extent to which the equal spending ratio result is violated across these settings, though it appears to be violated somehow less in close elections and in elections with more symmetric budgets.

D.2 Spending Growth

We now look at rates of spending growth. Again taking $\Delta = 1$ week, the consecutive period spending ratio (CPSR) is defined as x_{t+1}/x_t for the Democrat and y_{t+1}/y_t for the Republican candidate, which is defined for ten consecutive week pairs.⁴⁵ If the constant spending growth result of Section 2.3 holds, then these will be relatively stable over time. However, since there are candidates who spend zero in some of the earlier weeks, this ratio cannot be calculated for certain periods.

Given this, we calculate CPSRs using two approaches: (i) dropping all elections with zero spending in any week, and (ii) dropping all pairs of consecutive weeks that would include a week with zero spending.⁴⁶ These constitute two rules for focusing on different subsets of data. Approach (i) leaves us with only 221 (out of the total 601) elections in our

⁴⁵Recall that we drop the final incomplete week.

⁴⁶If zero spending occurs at time t , both x_{t+1}/x_t and x_t/x_{t-1} are excluded.

dataset where no zero spending occurs, and in approach (ii) we drop 1,692 consecutive week pairs out of a total of 13,222, which is only 12.8% of consecutive week pairs. Moreover, there is no instance of zero spending following positive spending in the sample: once a candidate starts spending a positive amount, she continues to do so until the election.

The distribution of average CPSRs for every candidate, along with their 95% confidence intervals from each of the two approaches are depicted in Figure D.2. The distributions obtained from approaches (i) and (ii) are very similar, as are the confidence intervals. The reported CPSR’s for the second approach can be interpreted as growth rates conditional on having started positive spending during an electoral campaign. This approach uses all of the available sample in getting estimates of CPSR and discards less data, hence we proceed with analyzing the growth rates obtained using the second method. Hereafter, when we say “growth rates,” we will be referring to growth rates conditional on having started spending positive amounts.

Since the tractable specifications of our model predict a stable growth rate in spending over time, we focus on empirically reporting how CPSRs change over the course of an election. Table D.2 displays a specific measure of central tendency for CPSRs: the share of candidates that remained within half a standard deviation of their election’s CPSR means, for every week in our dataset.

Looking at Figure D.2, the majority of the candidates in our dataset have relatively stable growth rates in spending over time. The middle 90% of the distribution of CPSR values (i.e. the 5th to 95th percentile) spans [0.98, 1.9]. For the candidate with the median value, we get an average CPSR of $\mu_r = 1.16$, meaning that their spending increased by 16% on average every week after they started spending positive amounts. The majority of candidates have relatively low standard deviations, with 62% having a standard deviation of $\sigma_r \leq 1$ and 87% of candidates having $\sigma_r \leq 2$. Any variation in CPSR values is usually driven by only a few weeks of volatile growth, rather than volatility in the entire spending path. On average, candidates remain within half a standard deviation of their mean CPSR value for 5.25 weeks (out of the 10 weeks that we can calculate CPSR values). 62% of the candidates remain within this range for more than 5 weeks, and 43% for more than 6 weeks. On an average week, about 54% of the candidates are within this range.

Table D.2 shows that the constant spending growth prediction is violated to a smaller extent as the election approaches and candidates begin to spend more substantial amounts. It also shows that statewide races, which typically see larger amounts of money spent, generally have smaller/fewer violations than House races, although the differences are very small. For example, in the last eight weeks of the elections, the CPSRs remain within half a standard deviation of their means for each candidate in 62.1%, 61.8%, and 59.3% of Senate, gubernatorial and House candidates, respectively.

One possible explanation for these deviations from the mean is that candidates value money left over as in our extension, and the constant spending growth prediction fails. But

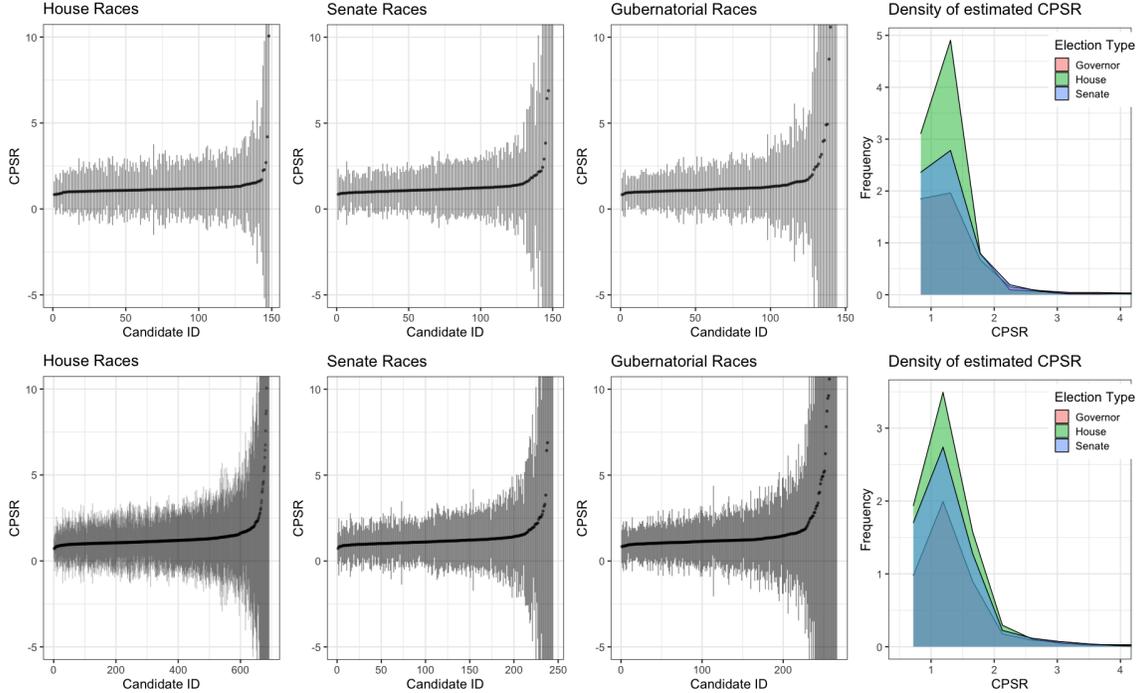


Figure D.2: Estimated CPSR values for candidates in our dataset, with 95% confidence intervals. The upper row are estimates of the CPSR that we get from dropping all elections with zero spending. The bottom row are estimates that we get from dropping all pairs of consecutive weeks that include zero spending. We also depict the densities of the CPSR across election types using both approaches.

if House candidates are more likely to value money left over than Senate or gubernatorial candidates (because the value of office is lower, or their future political ambitions—perhaps to become Senators or governors—are greater, or because they compete more frequently in future elections) this appears to be reflected only to a limited extent in the disaggregation based on the election type. Note that we also do not see major differences in the violation of the constant spending growth result across races that have early voting and those that do not. Thus, violations of equal spending growth are unlikely to be driven by the mechanism behind our early voting extension in which this prediction does not hold. Finally, a third possible explanation is that the candidates have uncertain budgets that react to their polling performance, as in our evolving budgets extension. Unfortunately, however, we cannot investigate whether the equilibrium spending path predicted by our evolving budget extension could account for these violations since data on when candidates receive money or pledges from donors are not available.

We also look at the extent of violations of the constant spending growth prediction in the other disaggregations that we looked at with spending ratios in Section D.1. Again we find very small differences across the different settings.

Table D.2: Consecutive Period Spending Ratios

	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	Overall
$\% \in (\mu_r - 0.5\sigma_r, \mu_r + 0.5\sigma_r)$	0.469	0.480	0.450	0.509	0.547	0.628	0.632	0.699	0.688	0.683	0.542
Senate	0.461	0.465	0.458	0.496	0.558	0.664	0.664	0.725	0.680	0.730	0.554
Governor	0.529	0.483	0.463	0.516	0.558	0.673	0.635	0.741	0.711	0.650	0.559
House	0.429	0.489	0.440	0.511	0.538	0.600	0.620	0.675	0.684	0.679	0.532
Early Voting	0.485	0.511	0.467	0.518	0.557	0.630	0.617	0.698	0.685	0.675	0.544
No Early Voting	0.441	0.421	0.418	0.491	0.529	0.627	0.662	0.703	0.696	0.699	0.539
Open Seat	0.462	0.447	0.448	0.474	0.545	0.629	0.625	0.685	0.719	0.690	0.536
Incumbent Competing	0.475	0.507	0.452	0.532	0.548	0.629	0.637	0.708	0.671	0.679	0.546
Close Election	0.455	0.476	0.462	0.549	0.571	0.676	0.656	0.709	0.688	0.691	0.552
Not Close Election	0.476	0.483	0.446	0.492	0.537	0.610	0.623	0.696	0.689	0.680	0.538
Close Budgets	0.490	0.487	0.501	0.528	0.546	0.656	0.662	0.722	0.759	0.726	0.566
Not Close Budgets	0.451	0.474	0.405	0.492	0.548	0.607	0.609	0.682	0.633	0.649	0.523

Note: The table reports the share of candidates for which the CPSRs are less than 0.5 standard deviations away from that candidate’s average CPSR over 11 weeks. Week -2 is missing because the final week is not included in the analysis. See the note under Table D.1 for the definition of close elections and close budgets.

E Additional Empirical Results

E.1 Results with 20 weeks

We replicate the estimation in the main text using data starting from 20 weeks before election day and including all the elections for which the two candidates spent positive amounts on TV advertising for at least two weeks during the course of the campaign. This gives us 1163 elections, instead of 601 elections as in the main text. We conduct the same exercise using our hierarchical model, except for a proper change of scale for the probability of observing zero variable, ϱ_{type} . Since the time can range from 19 to 1, ϱ_{type} is bounded between 0 and 0.052, and we set our uniform prior over the same interval for all three election types, Senate, House and gubernatorial.

The posterior values are reported in Table E.1. Overall, the distributions of estimated decay rates are very close. The estimation procedure yields approximately the same overall distribution for weekly decay rates regardless of the length of the period analyzed. This is because our model decomposes the zero generating process and the spending process. In Figure E.1 we plot the distributions of the estimates of perceived decay rates for the 12 and 20 week analyses side-by-side. In Figure E.2 we also plot the difference in median decay rate values calculated using 12 weeks and 20 weeks. For the vast majority of the elections, the difference in weekly decay rates is smaller than 0.05. For elections in which the additional 8 weeks carry information about the spending process and are not just informing the zero-

Parameter	\hat{R}	n_{eff}	mean	s.d.	2.5%	50%	97.5%
β_{O}	1.001	13034	0.225	0.085	0.059	0.225	0.391
β_{E}	1.001	10315	-0.110	0.084	-0.274	-0.110	0.056
β_{S}	1.000	10236	-2.565	0.111	-2.791	-2.563	-2.350
β_{H}	1.001	4727	-3.041	0.089	-3.222	-3.038	-2.872
β_{G}	1.000	10486	-2.529	0.112	-2.754	-2.528	-2.314
ϱ_{S}	1.000	49397	0.001	0.001	0.000	0.001	0.002
ϱ_{H}	1.000	50270	0.002	0.000	0.002	0.002	0.002
ϱ_{G}	1.000	44848	0.001	0.001	0.000	0.001	0.002
σ_{spend}^2	1.000	32085	0.011	0.000	0.011	0.011	0.011
σ_{λ}^2	1.002	2500	0.616	0.077	0.475	0.612	0.777

Table E.1: Model parameters with convergence diagnostics and posterior quantiles for 1163 elections and 20 weeks of data.

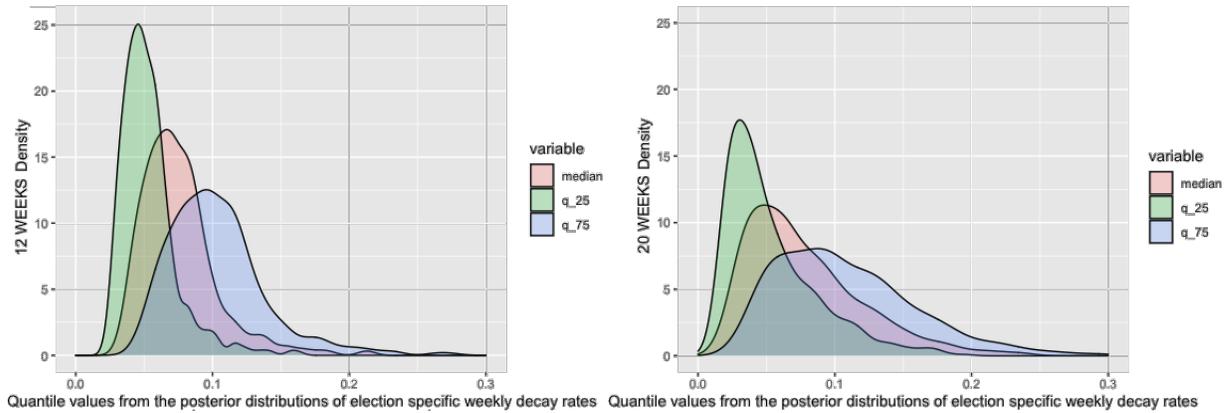


Figure E.1: Perceived decay rates, estimated using 12 and 20 weeks of spending data.

generating process, the decay rate estimates reflect this change. On average the difference is very small. This is expected, given that most (though, not all) of the spending levels between week 20 and week 12 are zero.

E.2 Estimates from the Evolving Budgets Model

The extension of our model in Section 3.4 allows for leads in popularity to affect the budget ratios. We estimate election specific weekly decay rates using this model.

To fix ideas and understand how leads in popularity might affect the budget ratios next period consider the following numerical example, where we use the notation from Section 3.4.

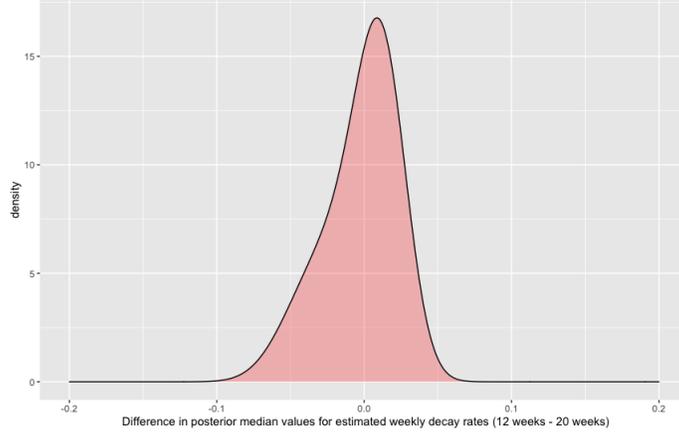


Figure E.2: The density of the difference between median posterior values for election specific decay rates, comparing the posteriors for 12 weeks of data and 20 weeks of data.

Let $\sigma_X^2 = \sigma_Y^2 = 0.1$, and let the covariance of the budget process be $\rho = 0.1$. Then we have

$$\mathbb{E} \left(\frac{X_{t+1}/Y_{t+1}}{X_t/Y_t} \right) = e^{(a-b)z_t + (1-0.1)\sigma_Y^2}.$$

Therefore, with a feedback parameter of say $a - b = 0.25$, a 10 percentage point lead in popularity for candidate 1 in a given week implies in the following week, the budget ratio will increase by about 12.2% in favor of the candidate.

The estimation model remains identical to the one in the main text, except that now we allow the mean of the spending to depend on $a - b$, thus denoting it $\mu(\lambda_j, t, a - b)$. The parameter $a - b$ from the extension determines how next period budget ratios depend on this period's popularity leads. Since $a - b$ and λ cannot be separately identified, we do not set a prior for $a - b$, but instead we use a grid of different values for $a - b$ and estimate the decay rates, showing how the values change under different regimes of budget feedback.

The budget feedback process parameterized by $a - b$ implies that candidates now have an additional incentive to spend early: any lead in popularity in a period implies a larger budget in the next period. Thus, if the feedback is positive, spending in early periods can be due either to low decay rates, or to the incentive to build a popularity lead and get more money. The model therefore attributes part of the observed early spending to this added incentive to build up a war chest.

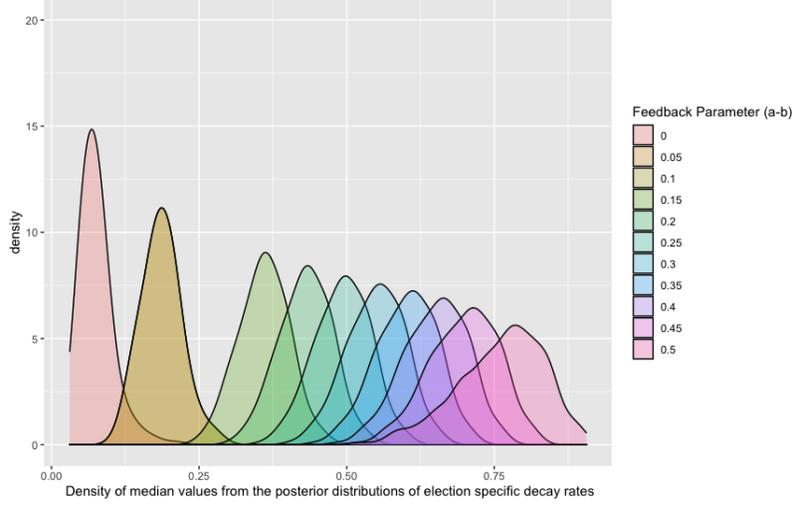


Figure E.3: The distribution of election specific weekly decay rates, estimated using 12 weeks of spending data and different values of the feedback parameter $a - b$.

Setting the expected spending in week t equal to the expected spending ratio for this extension in equation (19), i.e.

$$\mu(\lambda_j, t, a - b) = \frac{g_1(m - 1)}{g_1(m - 1) + g_2(m - 1) \frac{\lambda_j}{1 - e^{-\lambda_j \Delta}}},$$

we can estimate λ_j provided that we specify a value for $a - b$. In doing this, we keep assuming that $\Delta = 1$. To understand how implied decay rate estimates respond to budget feedback, we set $a - b$ equal to a grid of values between 0 and 0.5 and get posteriors from our hierarchical Bayes model.

The distributions of the perceived decay rates from this estimation are depicted in Figure E.3. Overall, we see that under zero feedback, the perceived weekly decay rates which best fit the data are quite low. Furthermore, they rise steadily with $a - b$. The estimated decay rates get close to those obtained by the literature through survey data for values of the budget feedback $a - b$ equal to 0.3 or larger. Hence, part of the discrepancy between the perceived decay rates that we estimate and the actual decay rates obtained by the literature could be explained by positive feedback in the budget process. However, whether these values of $a - b$ are justified remains an open question that can only be answered once the Federal Election Commission starts reporting budget process data.

Senate Elections in our Baseline Sample

Year	State
2000	DE, FL, IN, ME, MI, MN, MO, NE, NV, NY, PA, RI, VA, WA
2002	AL, AR, CO, GA, IA, LA, ME, NC, NH, NJ, OK, OR, SC, TN, TX
2004	CO, FL, GA, KY, LA, NC, OK, PA, SC, WA
2006	AZ, MD, MI, MO, NE, OH, PA, RI, TN, VA, WA, WV
2008	AK, CO, GA, ID, KS, KY, LA, ME, MS, NC, NE, NH, NM, OK, OR, SD
2010	AL, AR, CA, CO, CT, IA, IL, IN, KY, LA, MD, MO, NH, NV, NY, OR, PA, VT, WA
2012	AZ, CT, FL, HI, IN, MA, MO, MT, ND, NE, NM, NV, OH, PA, RI, VA, WI, WV
2014	AK, AR, CO, GA, IA, IL, KY, LA, ME, MI, MT, NC, NH, NM, OR, SD, VA, WV

Gubernatorial Elections in our Baseline Sample

Year	State
2000	IN, MO, NC, NH, WA, WV
2002	AL, AR, AZ, CA, CT, FL, GA, HI, IA, IL, KS, MA, MD, ME, MI, NM, NY, OK, OR, PA, RI, SC, TN, TX, WI
2004	IN, MO, NC, NH, UT, VT, WA
2006	AL, AR, AZ, CO, CT, FL, GA, IA, IL, KS, MD, ME, MI, MN, NH, NV, NY, OH, OR, PA, RI, TN, VT, WI
2008	IN, MO, NC, WA
2010	AK, AL, AR, AZ, CA, CT, FL, GA, HI, IA, ID, IL, MA, MD, MI, MN, NH, NM, NV, NY, OH, OK, OR, PA, SC, SD, TN, TX, UT, VT, WI
2012	IN, MO, MT, NC, ND, NH, WA, WV
2014	AL, AR, AZ, CO, CT, FL, GA, HI, IA, ID, IL, KS, MA, MD, ME, MI, MN, NE, NH, NM, NY, OH, OK, OR, PA, SC, TX, WI

House Elections in our Baseline Sample

Year	State-District
2000	AL-4, AR-4, CA-20, CA-49, CO-6, CT-5, FL-12, FL-22, FL-8, GA-7, KS-3, KY-3, KY-6, MI-8, MN-6, MO-2, MO-3, MO-6, NC-11, NC-8, NH-1, NH-2, NM-1, NV-1, OH-1, OH-12, OK-2, PA-10, PA-13, PA-4, TX-25, UT-2, VA-2, WA-1, WA-5, WV-2
2002	AL-1, AL-3, AR-4, CT-5, FL-22, IA-1, IA-2, IA-3, IA-4, IL-19, IN-2, KS-3, KS-4, KY-3, ME-2, MI-9, MS-3, NH-1, NH-2, NM-1, NM-2, OK-4, PA-11, PA-17, SC-3, TX-11, UT-2, WV-2
2004	CA-20, CO-3, CT-2, CT-4, FL-13, GA-12, IA-3, IN-8, KS-3, KY-3, MO-5, MO-6, NC-11, NE-2, NM-1, NM-2, NV-3, NY-27, OK-2, OR-1, TX-17, WA-5, WV-2
2006	AZ-5, AZ-8, CO-4, CO-7, CT-2, CT-4, CT-5, FL-13, FL-22, GA-12, HI-2, IA-1, IA-3, ID-1, IL-6, IN-2, IN-8, IN-9, KY-2, KY-3, KY-4, MN-6, NC-11, NH-2, NM-1, NV-3, NY-20, NY-24, NY-25, NY-29, OH-1, OH-12, OH-15, OH-18, OR-5, PA-10, SC-5, TX-17, VA-2, VA-5, VT-1, WA-5, WI-8
2008	AK-1, AL-2, AL-3, AL-5, AZ-3, AZ-5, AZ-8, CA-11, CA-4, CO-4, CT-4, CT-5, FL-16, FL-24, FL-8, GA-8, ID-1, IL-10, IN-3, KY-2, KY-3, LA-4, LA-6, MD-1, MI-7, MO-6, NC-8, NH-1, NH-2, NM-1, NM-2, NV-2, NV-3, NY-20, NY-24, NY-25, NY-26, NY-29, OH-1, OH-15, PA-10, PA-11, SC-1, VA-2, VA-5, WI-8, WV-2
2010	AL-2, AL-5, AR-2, AZ-1, AZ-5, AZ-8, CA-20, CA-45, CO-3, CO-4, CT-4, CT-5, FL-2, FL-22, FL-24, FL-8, GA-12, GA-8, HI-1, IA-1, IA-2, IA-3, IN-2, IN-8, KS-4, KY-6, MA-1, MD-1, MD-2, MI-1, MI-3, MI-7, MI-9, MN-6, MO-3, MO-4, MO-8, MS-1, NC-2, NC-5, NC-8, NE-2, NH-1, NH-2, NM-1, NM-2, NV-3, NY-20, NY-23, NY-24, NY-25, OH-1, OH-12, OH-13, OH-15, OH-16, OH-9, OK-5, OR-3, OR-5, PA-10, PA-11, PA-4, SC-2, SC-5, SD-1, TN-1, TN-4, TN-8, TN-9, TX-17, VA-2, VA-5, VA-9, WA-2, WI-8, WV-3
2012	AZ-2, CA-10, CA-24, CA-3, CA-36, CA-52, CA-7, CA-9, CO-3, CO-6, CO-7, CT-5, FL-18, GA-12, HI-1, IA-1, IA-2, IA-3, IA-4, IL-12, IL-13, IL-17, IL-8, IN-2, IN-8, KY-6, MA-6, ME-2, MI-6, MN-6, MN-8, MT-1, NC-7, ND-1, NH-1, NH-2, NM-1, NV-3, NY-19, NY-21, NY-24, NY-25, NY-27, OH-16, OH-6, PA-12, RI-1, SD-1, TX-23, UT-4, VA-2, VA-5, WI-8, WV-3
2014	AR-2, AZ-1, AZ-2, CA-21, CA-36, CA-52, CA-7, CO-6, CT-5, FL-18, FL-2, FL-26, GA-12, HI-1, IA-1, IA-2, IA-3, IL-10, IL-12, IL-13, IL-17, IN-2, ME-2, MI-7, MN-7, MN-8, MT-1, ND-1, NE-2, NH-2, NM-2, NV-3, NY-19, NY-21, NY-23, NY-24, VA-10, VA-2

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